APPENDIX,

OR,

SUPPLEMENT

TO THE

Treatiſe of Artillery:

CONTAINING

The true Projectile deſcribed by Bodies in the Air.
The greateſt Velocity and Reſiſtance they can have.
The moſt advantageous Length of Guns, and their Charges which produce the greateſt Effects; with an Introduction of Fluxions.
To which is added, the true Figure of the Earth, deduced from actual Menſuration.

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M DCC LXVIII.

PREFACE.

THE improvements in artillery and gunnery are the chief aim of this work; but as this could not be done without the higheſt and moſt intricate parts of the mathematics; and as ſeveral propoſitions, which are abſolutely neceſſary in what follows, are not to be found elſewhere; I was induced to begin with a conciſe treatiſe of fluxions; deduced from the principles laid down by Sir Iſaac Newton, the inventor, in his Introduction to the Quadrature of Curves. By this the reader will find every thing that is wanting to underſtand the ſubject, without being obliged to have recourſe to other authors.

In a ſubject ſo intricate as this, it is unſafe to depend on what has been ſaid by others without a very ſtrict examination, to avoid inaccuracies that might miſlead and occaſion wrong concluſions. The want of this precaution has been the cauſe that writers upon this ſubject, have not ſucceeded ſo well as might have been expected from their knowledge in other reſpects.

This work is the production of many years labour, as the great number of capital errors committed by others in their theories required time and application to diſcover. Thus the ſpecific gravity of air, as commonly ſuppoſed, we found not to be true: the common eſtimation of the reſiſtance of bodies moving in the air is quite void of truth, at leaſt as Mr. Robins will have it; for he makes that of a 24 pound ball above 23 times more than it poſſibly can be, as we have proved in this work. Again, the greateſt velocity a body can poſſibly acquire by falling in the air, has not been rightly determined, though it is abſolutely neceſſary to be

known, before projectiles in the air can be determined.

Seeing that the greateſt velocity that a ſhot can poſſibly have depends on its diameter, I was enabled to determine the moſt advantageous length of guns and their beſt charges (a queſtion ſought for in vain, ever ſince their invention) and to correct the vulgar error of making ſmall calibers longer, and loading them with more powder, in proportion to the large, in order, as they imagine, to make their ſhot go farther; whereby they were obliged to make them heavier than they need be, at a much greater expence in metal and carriage.

From what has been ſaid in this work, artillery may be brought to its greateſt perfection by a few good experiments: for all thoſe that have hitherto been publiſhed, I may ſay, are trifling, and not the leaſt to the purpoſe: the only good are thoſe made by General Williamſon, Major Hiſlop, and ſeveral other artillery officers at Minorca, in 1745; which they were ſo kind to communicate to me, and are inſerted in the introduction to our artillery.

The uſe of fluxions, in the improvement of artillery and gunnery, may poſſibly ſtartle many artillery officers, but as it cannot be done otherwiſe, I cannot help it: though an officer may underſtand his buſineſs very well in all other reſpects, without ſo complex a theory as this, yet ſome of them will be pleaſed to find how they may improve their knowledge; the gallant behaviour of all thoſe employed in the two laſt wars has been ſo conſpicuous to all the world that they need not be under any apprehenſion, that the utmoſt malice or ſpite could invent can be the leaſt detraction of their merit.

Though the figure of the earth has no immediate connection with what we have treated before, yet I hope the curious reader will not be diſpleaſed to ſee it treated in an eaſy and familiar manner.

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THE FIRST PRINCIPLES OF FLUXIONS.

SECTION I.

Of FLUXIONS *in General.*

THOUGH Fluxions have been treated by many able authors, and their application in a very copious manner; yet, if I am not miſtaken, the firſt principles have not hitherto been explained in ſo eaſy and perſpicuous a manner as might be done. Two different methods are uſed to demonſtrate them; the one by the laws of motion, and the other by the prime and ultimate ratios. The firſt requires more previous knowledge, and is not ſo applicable in many caſes; for inſtance, in finding the fluxion of a curve line, or that of its ſurface, deſcribed about an axis, as any one may be convinced who reads *Maclaurin*'s work upon this ſubject. The ſecond is that which has been uſed by the inventor Sir *Iſaac Newton*, as may be ſeen in the introduction to his quadrature of curves. But thoſe authors who uſe this method, have generally miſtaken the limiting ratio for the prime and ultimate ones, ſuch as Mr. *Robins* and many others, which has occaſioned ſo much diſputes about the certainty of their principles.

In the method of fluxions, quantities are not conſidered as made up by the accumulation of parts, but as generated by a continual flux or motion; that is, a line deſcribed by a point, a ſurface by a line, and a ſolid by a ſurface.

DEFINITION I.

The velocities or rates with which quantities, varying by a continual motion, increaſe or decreaſe at the ſame time, are called *fluxions*, and the quantities themſelves *fluents*.

DEFINITION II.

The ultimate *ratio* of variable quantities, which vaniſh or ceaſe to exiſt at the ſame time, is the laſt ratio they can have, and in which they vaniſh *; and the prime ratio, that in which quantities enter together into exiſtence.

Sir *Iſaac Newton* obſerves, that the fluxions of variable quantities, are as the ultimate ratios of their cotemporary increments. But as this inference may not readily appear to beginners, we ſhall endeavour to prove it in the two following theorems.

THEOREM. *Plate* I. *Fig.* 1.

1. *The difference* mn *or* pP, *between two abſciſſas* AP, Ap, *of a curve* AM, *and the difference* nM *between the*

* This definition of the ultimate ratio is ſo exact and preciſe, that one ſhould think it impoſſible to miſtake its meaning; ſince it requires that the quantities ſhould abſolutely vaniſh at the ſame time, and therefore all ſuch quantities as do or cannot actually vaniſh, tho' they approach ever ſo near, can have no ultimate ratio: yet the limiting ratio, which may approach ever ſo near, but the quantities can never vaniſh, has been confounded with the ultimate ratio. To ſhew the difference between theſe two ratios, let a be a conſtant, and x a variable quantity; then it is ſaid, that the ratio of a to $a+x$, becomes that of equality, when x becomes infinitely ſmall or leſs than any aſſignable quantity. Is not this ſuppoſing $a=a+x$? It is in vain to ſay that x can make no difference on account of its ſmallneſs; it muſt be either ſomething or nothing, ſince there can be no ſtate between exiſtence and non-exiſtence; and if x be ſomething, tho' ever ſo ſmall, a can never be equal to $a+x$, in a mathematical ſenſe. Tho' this ſuppoſition were admitted, it muſt be owned, that, to eſtabliſh the principles of ſo extenſive a ſcience as fluxions upon this foundation, muſt appear very improper, as it introduces a quantity of which we can have no poſſible idea; and, what is ſtill more abſurd, and againſt good method, when there is not the leaſt occaſion for it.

corre-

correſponding ordinates PM, pm, *vaniſh in the ratio of the ſubtangent* TP *to the ordinate* PM.

Let the line drawn thro' the extremities M, m of the ordinates meet TA in V; then the ſimilar triangles VPM, mnM, give mn : nM :: VP : PM; and as this proportion always ſubſiſts, where-ever the ordinate pm may be drawn, if we ſuppoſe pm to move parallel to itſelf towards PM, it is plain, that as pm approaches PM, or the point p to the point P, the point V approaches alſo to the point T, and the lines mn, nM diminiſh gradually till the point p coincides with the point P, then they vaniſh together, and the point V in that inſtant, and not before nor after, coincides with the point T: conſequently, by the proportion above, the ultimate or vaniſhing ratio of mn to nM, will be equal to the ratio of the ſubtangent TP to the ordinate PM.

N. B. As but one line can be drawn ſo as to touch a curve in the ſame point, ſo can there be but one ultimate ratio between the differences of the increments or decrements of the abſciſſas and ordinates of the ſame curve.

It may alſo be obſerved, that ſince mn : mM :: VP : VM, the ultimate ratio of mn to the arc mM is equal to the ratio of the ſubtangent TP to the tangent TM; for if a tangent be drawn thro' the point m, it is eaſily proved that the ſum of the tangents at m and M, terminated by their interſection, will always be greater, and the chord mM leſs, than the arc mM; and ſince the ſum of theſe tangents and the chord mM vaniſh in the ratio of equality, the arc mM and that chord vaniſh alſo in the ſame ratio: conſequently the ultimate ratio of mn to mM is equal to the ratio of the ſubtangent TP to the tangent TM.

THEOREM. *Fig.* 2.

2. *If a line* pE *moves parallel to itſelf with an uniform motion, while a point* m *in that line moves with a*

retarded one ſo as to deſcribe a curve AM; *when that line arrives into any poſition* PM, *the velocity of the point* p *is to the velocity of the point* m, *as the ſubtangent is to the ordinate.*

Let TP be taken to PM, as the velocity of the point p with which it arrives at P, is to the velocity the point m arrives with at M; thro' the point M draw EF parallel to AP, and let the ordinates pm, qn drawn on both ſides of PM, meet EF, in E, F and TM, in L, N: Then as the velocity of the point m is greater at m and leſs at n than at M, by ſuppoſition, while the line pE moves from p to P, and from P to q, or the point E from E to M, and from M to F, uniformly, the point m which deſcribes the curve, moves over mE with a greater and over Fn with a leſs velocity than that point has at M; and ſince (TP : PM : :) EM : EL : : MF : FN; by ſuppoſition the ſpaces EL, FN, are deſcribed uniformly with a velocity equal to that which the point m has at M, at the ſame time that EM, MF are deſcribed uniformly, with the velocity of the point p; and as mE is deſcribed with a greater velocity, and Fn with a leſs, than that with which LE and FN are deſcribed at the ſame time, LE will be leſs than Em, and FN greater than Fn. The points L, N, are therefore on the outſide of the curve AM; and as this will always happen where-ever the ordinates pm, qn, are drawn, excepting at the point M, the line TM touches the curve at M. The velocity of the point p at P is conſequently to the velocity of the point m, at M, or the fluxion of the abſciſſa AP is to the fluxion of the ordinate PM, as the ſubtangent TP is to the ordinate PM.

If the motion of the point m was accelerated inſtead of being retarded, the curve would be convex towards the line AP; but the demonſtration will remain the ſame as before.

We have here ſuppoſed the common laws of motion, as being treated of in books of geometry, and

and therefore ſuppoſed to be known; but if any reader is unacquainted with them, he may paſs over this laſt theorem, as being only of uſe to ſhew the agreement between fluxions and ultimate ratios.

COR.

3. Hence, the ultimate ratio of the cotemporary increments or decrements of the abſciſſa AP and ordinate PM of any curve is then the ſame, or equal to the ratio of the fluxions of theſe lines, agreeable to Sir *Iſaac Newton*'s aſſertion: the demonſtrations deduced from the ultimate ratios are conſequently equally concluſive and geometrical, as thoſe from fluxions, and in many caſes more conciſe and convenient, as will appear hereafter.

Obſerve, that no actual motion is required in the method of fluxions, but only the proportion, whereby variable quantities increaſe or decreaſe at the ſame time, which is always aſſignable from the known relation between the quantities themſelves. The word *velocity* is applied in a figurative ſenſe, to denote the degree of increaſe and decreaſe at the ſame time.

COR.

4. Since the direction of the motion in any point of the curve is always in the tangent at that point, and if the motion with which the point m arrives at M, was to become uniform, the point m would proceed in the direction of the tangent TM; the directions of the motions in the abſciſſa AP, ordinate PM, and the curve, being in the ſubtangent TP, ordinate PM and tangent TM; the velocities, or fluxions of the abſciſſa, ordinate and curve, may therefore be repreſented by the three ſides of the triangle TPM, or by thoſe of any other triangle ſimilar to this.

Fig. 3. No. 2. Hence if AM be an arc of a circle, O the center, PM the ſine: then becauſe of the ſimilar right angled triangles OPM, MPT, we have, 1°. OP: OM::PM:MT, that is, the coſine is to the radius as the fluxion of the ſine is to the fluxion of the arc AM.

2°. PM:OM::PT:MT, that is, the ſine is to the radius as the fluxion of the coſine is to the fluxion of the arc AM.

N.B. Conſtant or invariable quantities have no fluxions, and are generally expreſſed by the initial letters a, b, c, d of the alphabet; the variable or flowing ones, by the finals v, x, y, z, and their fluxions by the ſame letters with a point over them, as $\dot{v}, \dot{x}, \dot{y}, \dot{z}$.

AXIOM.

Equal magnitudes have equal fluxions, ſince they muſt neceſſarily increaſe or decreaſe at an equal rate at the ſame time.

THEOREM.

5. *If the ſum of ſeveral magnitudes is equal to one or more magnitudes, the ſum of their fluxions will be equal to the fluxions of their equals.*

This is evident from the axiom; for if $v+y=x+z$, then will $\dot{v}+\dot{y}=\dot{x}+\dot{z}$, otherwiſe theſe magnitudes could not increaſe or decreaſe at an equal rate.

THEOREM.

6. *When a magnitude decreaſes while others increaſe, its fluxion muſt be negative.*

For if $y+z=a$, and a is a conſtant magnitude; then when y increaſes z muſt decreaſe, or if y decreaſes z muſt increaſe: and ſince the fluxion of $y+z=a$ is $\dot{y}+\dot{z}=o$, becauſe a is conſtant, we have $-\dot{y}=\dot{z}$, or $\dot{y}=-\dot{z}$.

THEOREM.

7. *If variable magnitudes are in a conſtant ratio, their fluxions will be in the ſame ratio.*

For ſince variable magnitudes, which increaſe or decreaſe in a conſtant ratio, their rates or fluxions muſt be in the ſame ratio: thus, if a, b are conſtant, and x, y variable, and if $a : b :: x : y$, then x and y muſt vaniſh together, and in the ratio of a to b.

PRO-

PROBLEM. *Fig.* 1.

8. *To find the fluxion of a variable ſquare,* yy.

Let y expreſs the ordinate PM of a parabola whoſe parameter is unity, and abſciſſa AP$=x$; then will $x=yy$, and becauſe the ſubtangent TP is double the abſciſſa AP by conicks, and the fluxion of the abſciſſa AP is to that of the ordinate PM, as TP is to PM, by *Art.* 2. we have TP $(2x)$: PM (y) :: $\dot{x}:\dot{y}$, or $2x\dot{y}=y\dot{x}$, or becauſe $x=yy$, by writing the value of x into $2x\dot{y}$, we get $2\dot{y}y^2=y\dot{x}$, which divided by y gives $2\dot{y}y=\dot{x}$; but $\dot{x}$ is the fluxion of x, therefore $2\dot{y}y$ is that of its equal yy by the axiom.

Otherwiſe by the ultimate ratios. *Fig.* 1.

Let Ap$=u$, pm$=z$; then will $u=zz$, by the property of the parabola; this equality ſubtracted from $x=yy$ gives $x-u=yy-zz$; whence mn $(x-u)$: nM $(y-z)$::$y+z$:1; and by the ſimilarity of triangles mn:nM::VP:PM; therefore $y+z$:1::VP:PM, by equality of ratios. Now, when Ap becomes AP and and pm becomes PM, or $u=x$ and $y=z$, then VP becomes the ſubtangent TP, and conſequently $2y$: 1:: (TP : PM::) $\dot{x}:\dot{y}$, by *Art.* 1. or $\dot{x}=2\dot{y}y$.

COR.

9. Hence, the fluxion of a variable rectangle yz is $\dot{y}z+y\dot{z}$. For if we ſuppoſe $y+z=x$, then will $\dot{y}+\dot{z}=\dot{x}$ by *Art.* 5. and $y+z=x$ multiplied by $\dot{y}+\dot{z}=\dot{x}$, gives $y\dot{y}+\dot{y}z+y\dot{z}+z\dot{z}=x\dot{x}$, and the ſquare of $y+z=x$ is $yy+2yz+zz=xx$. Hence, ſince $y\dot{y}$, $z\dot{z}$, $x\dot{x}$, are half the fluxions of the ſquares yy, zz, xx, by *Art.* 8. it is manifeſt that $\dot{y}z+y\dot{z}$ is half the fluxion of $2zy$, that is of yz by the axiom.

COR.

10. The fluxion of y^3 is $3\dot{y}y^2$. For ſuppoſe $x=yy$; then $xy=y^3$ by equal multiplication, and the fluxion $\dot{x}y+x\dot{y}$ of xy is equal to the fluxion of y^3 its equal, by the axiom; but the fluxion of $x=yy$, is $\dot{x}=2y\dot{y}$, by *Art.* 8, and the values of x, $\dot{x}$, being wrote into the

fluxion $\dot{x}y+x\dot{y}$, gives $2\dot{y}y^2+\dot{y}y^2$, or $3\dot{y}y^2$, for the fluxion of y^3.

The fluxion of y^4 is $4\dot{y}y^3$; for if $x=yy$, which ſquared gives $xx=y^4$, and the fluxion $2x\dot{x}$ of x^2 will be equal to the fluxion of y^4, by the axiom: now the fluxion of $x=yy$ is $\dot{x}=2y\dot{y}$, by *Art.* 8. and theſe values of x, $\dot{x}$, wrote into the fluxion $2x\dot{x}$, gives $4\dot{y}y^3$ for the fluxion of y^4.

In the ſame manner the fluxion of y^5 is found to be $5y^4$, by ſuppoſing $x=y^4$, and that of y^6 is $6\dot{y}y^5$, by making $x=y^5$: in general the fluxion of any quantity y^n, whoſe index n is any whole poſitive number, is $n\dot{y}y^{n-1}$, that is equal to the product of the index n multiplied by the fluxion $\dot{y}$ of y, and this product by the quantity y^n, whoſe index is diminiſhed by unity.

C O R.

11. The fluxion of yz^n is $\dot{y}z^n+ny\dot{z}z^{n-1}$. For if $x=z^n$, then $xy=yz^n$, and the fluxion $\dot{x}y+x\dot{y}$ of xy, by *Art.* 9, will be equal to the fluxion yz^n, by the axiom; but the fluxion of $x=z^n$, is $\dot{x}=n\dot{z}z^{n-1}$, by the laſt; and the value of x, $\dot{x}$ wrote into $\dot{x}y+x\dot{y}$, gives $\dot{y}z^n+ny\dot{z}z^{n-1}$, for the fluxion of $\dot{y}z^n$.

Hence the fluxion of y^{-n} is $-n\dot{y}y^{-n-1}$, for if $x=-y^n$, then $xy^n=1$, whoſe fluxion $\dot{x}y^n+n\dot{y}xy^{n-1}$ is $=o$, or dividing by y^n, $\dot{x}=-n\dot{y}xy^{-1}$; or becauſe $x=-y^n$, we got $\dot{x}=-n\dot{y}y^{-n-1}$.

C O R.

12. The fluxion of $y^{\frac{n}{r}}$ is $\frac{n}{r}\dot{y}y^{\frac{n}{r}-1}$. For if $x=y^{\frac{n}{r}}$; then by raiſing both ſides of the equation to the power r, $x^r=y^n$; whoſe fluxion is $r\dot{x}x^{r-1}=n\dot{y}y^{n-1}$: the firſt ſide being divided by $\frac{r}{x}x^r$, and the ſecond by its equal $\frac{r}{x}y^n$, gives $\dot{x}=\frac{n}{r}x\dot{y}y^{-1}$, or becauſe $x=y^{\frac{n}{r}}$, we get $\dot{x}=\frac{n}{r}\dot{y}y^{\frac{n}{r}-1}$ for the fluxion required, which

ſhews

ſhews *that the fluxion of* y^n *is always* $n\dot{y}y^{n-1}$, *whether the index* n *is a whole or broken rational number, poſitive or negative.*

COR.

13. The fluxion of xyz, is found by making $u=xy$; for then $uz=xyz$, and the fluxion $\dot{u}z+u\dot{z}$ of uz by *Art.* 9, is equal to the fluxion of xyz, by the axiom; but the fluxion of $u=xy$ is $\dot{u}=\dot{x}y+x\dot{y}$, and theſe values of u and $\dot{u}$, being wrote into the fluxion $\dot{u}z+u\dot{z}$, gives $\dot{x}yz+x\dot{y}z+xy\dot{z}$ for that required. Or if $xyz=u$, then the fluxion $\dot{x}yz+x\dot{y}z+xy\dot{z}=\dot{u}$, divided, the firſt ſide by xyz, and the ſecond by its equal u, gives $\frac{\dot{x}}{x}+\frac{\dot{y}}{y}+\frac{\dot{z}}{z}=\frac{\dot{u}}{u}$.

COR.

14. The fluxion of $\overline{a+cz^n}^r$ is found by ſuppoſing $x=a+cz^n$, whoſe fluxion is $\dot{x}=nc\dot{z}z^{n-1}$, and $x=a+cz^n$ raiſed to the power $r-1$, and multiplied by r, gives $rx^{r-1}=r\times\overline{a+cz^n}^{r-1}$. The firſt ſide multiplied by $\dot{x}$, and the ſecond by its equal $nc\dot{z}z^{n-1}$, gives $r\dot{x}x^{r-1}=ncr\dot{z}z^{n-1}\times\overline{a+cz^n}^{r-1}$ for the fluxion of x^r or of its equal $\overline{a+cz^n}^r$.

GENERAL RULES.

I. *To find the fluxion of a ſingle quantity* y^n, *whoſe index* n *is any rational number.*

Multiply that quantity by the index n, and change one of its dimenſions into the fluxion $\dot{y}$.

II. *To find the fluxion of a binomial* $\overline{a+cz^n}^r$, *whoſe index* r *is any rational number.*

Multiply the fluxion of the binomial by the index r, and this product, by the binomial raiſed to a power whoſe index is unity, leſs than that of the propoſed one.

III. *To find the fluxion of any product* xyz, *of ſeveral variable quantities.*

Multiply the fluxions $\dot{x}$, $\dot{y}$, $\dot{z}$, of theſe quantities, each by the product of the others, yz, xz, xy, and the ſum

ſum of all theſe products will be the fluxion required.

EXAMPLES.

The fluxion of $au+bx^2+cy^3+dz^4$, is $a\dot{u}+2bx\dot{x}+3cy\dot{y}^2+4dz\dot{z}^3$, by *Rule* I.

The fluxion of $\sqrt{aa+xx}$, or $\overline{aa+xx}^{\frac{1}{2}}$, is $\frac{1}{2}\times 2x\dot{x}\times \overline{aa+xx}^{\frac{1}{2}-1}$, or $\frac{x\dot{x}}{\sqrt{aa+xx}}$ by *Rule* II.

The fluxion of $\sqrt{ax+xx}$ or $\overline{ax+xx}^{\frac{1}{2}}$, is $\frac{1}{2}\overline{a\dot{x}+x\dot{x}}\times\overline{ax+xx}^{\frac{1}{2}-1}$, or $\frac{a\dot{x}+2x\dot{x}}{2\sqrt{ax+xx}}$, by the ſame *Rule*.

And the fluxion of $x\sqrt{aa+xx}$, is $\dot{x}\sqrt{aa+xx}+\dot{x}x^2\times\overline{aa+xx}^{-\frac{1}{2}}$, by *Rule* III.

In the ſame manner may be found the fluxion of any multinomial, by following the ſame ſteps as in *Art.* 14. or that of a multinomial multiplied by any other quantity, by *Art.* 13.

THEOREM. *Fig.* 4.

15. *Whatever the magnitude may be which has but one variable dimenſion, its fluxion will always be in the compound ratio of the fluxion of the variable dimenſion, and of the conſtant dimenſions.*

This is manifeſt, becauſe the fluxion of ax is $a\dot{x}$, whatever conſtant dimenſions a may expreſs.

Thus if a given line PM, by moving along another line AD, parallel to itſelf, deſcribes a rectangle AC; the fluxion of any part ABMP, will be equal to the rectangle made by that line and the fluxion of the part AP, moved over by the point P. If the rectangle AC revolves about the baſe AD as an axis, ſo as to deſcribe a cylinder, the fluxion of any part deſcribed by the rectangle AM, will be equal to a cylinder whoſe baſe is the circle deſcribed by the given line PM, and altitude the fluxion of AP. In the ſame manner, the fluxion of the cylindric ſurface, deſcribed

ſcribed by the variable line BM, will be equal to the rectangle made by the circumference deſcribed by the radius PM, and the fluxion of the variable line BM, or that of its equal AP.

PROBLEM. *Fig.* 5.

16. *To find the fluxion of the area* AMP, *terminated by the curve* AM, *and the ordinate* PM, *perpendicular to the abſciſſa* AP.

Draw AB, of any given length, perpendicular, and BC parallel to AP, and let the ordinate QN, PM, produced meet BC in q and p: then if the ordinate mn, be ſuch, that the area QM be equal to the rectangle made by the ordinate mn and the baſe QP; the difference QM between the the areas AMP, ANQ, will be to the difference qP between the rectangle Ap, Aq as mn×QP is to Pp×QP, or as mn is to Pp. Now if the ordinate QN moves parallel to itſelf till it coincides with PM, it is evident that mn, which is always greater than QN, and leſs than PM, will likewiſe coincide with PM. The ultimate ratio of the area QM, to the rectangle Q p, or the fluxion of the area AMP to the fluxion of the rectangle Ap, is therefore as PM to Pp, by *Art.* 3. But the fluxion of the rectangle Ap is equal to the rectangle made by the given line Pp and the fluxion of AP, by *Art.* 15. the fluxion of the area AMP, is conſequently equal to the rectangle made by the ordinate PM and the fluxion of the abſciſſa AP.

PROBLEM.

17. *To find the fluxion of the ſolid deſcribed by the area* AMP, *about the axis* AP.

If the ordinate mn be ſuch, that the cylinder, whoſe baſe is the circle, deſcribed by that ordinate and altitude the line QP, be equal to the ſolid deſcribed by the ſpace QM: then the ſolid deſcribed by the ſpace QM will be to the cylinder, deſcribed by the rectangle Qp, about the ſame axis AP, as the circle deſcribed

by

by mn is to the circle described by Pp, and by a similar argument as in the laſt, the ultimate ratio of the ſolid deſcribed by the ſpace QM, to the cylinder deſcribed by the rectangle Qp, or the fluxion of the ſolid deſcribed by the area AMP, is to the fluxion of the cylinder deſcribed by the rectangle Ap, as the circle deſcribed by the ordinate PM is to the circle deſcribed by Pp. But the fluxion of the cylinder is equal to the cylinder whoſe baſe is the circle deſcribed by Pp and altitude the fluxion of AP, by *Art.* 15; the fluxion of the ſolid deſcribed by the area AMP, about the axis AP, is conſequently equal to the cylinder whoſe baſe is the circle deſcribed by the ordinate PM and altitude the fluxion of AP.

PROBLEM.

18. *To find the fluxion of the ſolid deſcribed by the area* AMP, *about the targent* AT *as an axis.*

If the ordinate mn be ſuch, that the ſolid deſcribed by the ſpace QM be equal to the cylinder whoſe ſurface is that deſcribed by mn, and baſe by Q'; then the ſolid deſcribed by the ſpace QM is to the ſolid deſcribed by the rectangle Qp, about the ſame axis, as the ſurface deſcribed by mn is to the ſurface deſcribed by Pp. Therefore, the ultimate ratio of the ſolid deſcribed by the area QM to the ſolid deſcribed by the rectangle Qp, or the fluxion of the ſolid deſcribed by the area AMP, is to the fluxion of the cylinder deſcribed by the rectangle Ap; as the ſurface deſcribed by the ordinate PM to the ſurface deſcribed by Pp; and ſince the fluxion of the cylinder is equal to the ſolid made by the ſurface deſcribed by Pp and the fluxion, of AP, by *Art.* 15, the fluxion of the ſolid deſcribed by the area AMP about the tangent AT is alſo equal to the ſolid made by the ſurface deſcribed by the ordinate PM and the fluxion of AP.

PRO

PROBLEM. *Fig.* 6.

19. *To find the fluxion of the ſurface deſcribed by the arc* AM, *of a curve about the axis* AP.

Draw DM parallel to AP, meeting the tangent at A in D, and the ordinate pm produced in n: then if c expreſſes the circumference of a circle deſcribed by the point M, and d another ſuch, that the rectangle made by it and the chord mM, may be equal to the ſurface deſcribed by that chord in the rotation of the figure about the axis AP; the cylindric ſurface deſcribed by nM will be to the conic ſurface deſcribed by the chord mM, as $c \times nM$ to $d \times mM$: and when Ap becomes AP, the circumference d becomes equal to c, and the ultimate ratio of the chord mM, or of the arc mM, by *Art.* 4. to nM, is equal to the ratio of the tangent TM to the ſubtangent TP. Conſequently the fluxion of the cylindric ſurface deſcribed by DM is to the fluxion of the ſurface deſcribed by the arc AM. as $c \times TP$ to $c \times TM$, by *Art.* 3. or as TP to TM; that is as the fluxion of the abſciſſa to the fluxion of the curve AM, by *Cor.* 4; and ſince $c \times TP$ expreſſes the fluxion of the cylindric ſurface, by *Art.* 15. $c \times TM$, will conſequently expreſs the fluxion of the ſurface deſcribed by the arc AM

Having thus given the fluxions of ſuch magnitudes as moſt commonly occur, we ſhall now inveſtigate ſome others no leſs uſeful than the former.

THEOREM. *Fig.* 7, 8.

20. *If in the tangent at* M *of a curve* AM, *the part* MT, *be taken for the fluxion of the arc* AM, *and a line* OM *be drawn from a given point* O *in the axis* AP *thro' the point* M, *the perpendicular* TR *to that line, will determine its fluxion* MR.

Draw the ordinate PM, and TV, perpendicular to it, draw likewiſe, Vr perpendicular to TR, and Vn to OM. If $OP = x$, $PM = y$, $OM = z$, then will $MV = \dot{y}$,

$=\dot{y}$, TV$=\dot{x}$, by *Art.* 4. and by the ſimilarity of the triangles OPM, VnM, we have OM (z) : PM (y) :: MV $(\dot{y})$: nM, or $z \times nM = y\dot{y}$; by the ſimilarity of the triangles OPM, VrT, OM(z) : OP(x) :: TV$(\dot{x})$: Vr, or $z \times Vr = x\dot{x}$; and ſince $Vr \pm nM = MR$, we have $z \times MR = x\dot{x} \pm y\dot{y}$; but the right angled triangle OPM, gives $zz = xx + yy$, whoſe fluxion is $z\dot{z} = \dot{x}\dot{x} \pm y\dot{y}$, according as y increaſes or decreaſes, while x increaſes, and therefore $z \times MR = z\dot{z}$, or $MR = \dot{z}$.

THEOREM.

21. *The fluxion of the ſector* OMA, *is equal to* $\frac{1}{2}$OM$\times$TR.

The ſimilar triangles OPM, VrT, give $z \times Tr = y\dot{x}$, and the ſimilar triangles OPM, VnM, $z \times Vn = x\dot{y}$; and ſince $Vn \mp Tr = TR$, we get $z \times TR = x\dot{y} \mp \dot{x}y$; now becauſe $y\dot{x}$ is the fluxion of the ſegment AMP, by *Art.* 16. and $\frac{1}{2}x\dot{y} \mp \frac{1}{2}y\dot{x}$, that of the triangle OPM$=\frac{1}{2}xy$, by *Art.* 9. the ſum or difference gives $\frac{1}{2}x\dot{y} \mp \frac{1}{2}y\dot{x}$ or its equal $\frac{1}{2}$OM$\times$TR, for the fluxion of the ſector OAM.

COR.

22. It is manifeſt that TR expreſſes the fluxion of the circular arc deſcribed from the center O with the radius OM, and whoſe ſine is PM. For if z be ſuppoſed conſtant, the fluxion of $zz = yy + xx$, will be $y\dot{y} \pm x\dot{x} = o$, and writing the value of $\dot{x}$ into $z \times TR = x\dot{y} \pm y\dot{x}$, we get $zx \times TR = \dot{y} \times \overline{yy + xx}$, or becauſe $zz = yy + xx$, $x \times TR = z\dot{y}$; and ſince the coſine OP (x) is to the radius QM (z) as MV to MT, or as the fluxion $\dot{y}$ of the ordinate PM is to the fluxion TR of the arc, by *Art.* 4, *No.* 2.

COR.

23. Hence, by drawing the chord AM, and producing the tangent MT, ſo as to meet the axis in Q; if PQ$=\dot{x}$, and PM$=\dot{y}$; the fluxion of the triangle OAM, will be $\frac{1}{2}$AQ$\times$PM, and that of the ſector OMA

OMA is OPM∓QPM, or $\frac{1}{2}$OQ×PM. Conſequently the fluxion of the triangle OAM is to the fluxion of the ſector OMA, as OA is to OQ; and the fluxion of the triangle OMA is to the fluxion of the ſegment AMA, as OA to OA—OQ or AQ.

COR. *Fig.* 9.

24. If from any other point C in the axis OA, a line CM be drawn to the ſame point M, and from the ſame point T in the tangent at M, the lines TR, TS, perpendicular to OM and CM, will expreſs the fluxions of the circular arcs deſcribed from centers O, C, thro' the point M, and whoſe ſines is PM, by *Art.* 22, and MR, MS, are the fluxions of the lines OM, CM, by *Art.* 20, when MT expreſſes the fluxion of curve AM.

COR. *Fig.* 10.

25. *Laſtly*, If from any given point O, in the axis OA, a line be drawn meeting the lines AM, BN, in M and N, and if in the tangents at M, N, the parts MT, NV, are taken for the fluxions of AM, BN, the perpendiculars TR, VX, to ON, will be proportional to the radii OM, ON; ſince they expreſs the fluxions of ſimilar circular arcs by *Art.* 22; and conſequently the fluxions of the ſpaces OAM, OBN, are as the ſquares of the lines OM, ON, by *Art.* 21.

Of various Orders of Fluxions.

When fluents are not augmented by any uniform velocity, it is convenient in many problems to conſider how theſe velocities vary. The rate of this variation is called the fluxion of fluxion, or the ſecond fluxion of the fluent. Theſe ſecond fluxions may alſo vary in different magnitudes of the fluent, and the rates of theſe variations is called the third fluxion. This may continue to as many ſubordinate degrees of fluxions, as the fluent admits of variations.

To give a clear and diſtinct notion of the different degrees

degrees of fluxions, it muſt be obſerved, that the fluxions of a curve, abſciſſa and ordinate, may be expreſſed by the three ſides of a triangle formed by the tangent, ſubtangent and ordinate, by *Art.* 4. and ſince the ſides of this triangle may vary in different points of the curve, the rates or theſe variations are called the ſecond fluxions; and as theſe variations may likewiſe be expreſſed by the ſides of another triangle, the fluxions of its ſides are called the third fluxions.

As one ſide of a triangle may be conſtant, without changing the rate of variation; ſo one of the fluxions may be ſuppoſed conſtant, while the others vary; the ratio of any two may likewiſe be ſuppoſed given, provided regard is had of this ſuppoſition in future operations.

The firſt fluxions, algebraically expreſſed, being marked with a point, the ſecond and third are likewiſe marked by two and three points. Thus the firſt fluxions of x, y, z, being marked thus, $\dot{x}$, $\dot{y}$, $\dot{z}$; the ſecond are $\ddot{x}$, $\ddot{y}$, $\ddot{z}$; and the third $\dddot{x}$, $\dddot{y}$, $\dddot{z}$.

The ſecond and third fluxions are found in the ſame manner as the firſt, by conſidering the former as variable magnitudes; thus, if $x = y^n$, denotes the equation of a curve, then will $\dot{x} = n\dot{y}y^{n-1}$, and if $\dot{y}$ be ſuppoſed conſtant, then will $\ddot{x} = n \times \overline{n-1}\dot{y}^2y^{n-2}$ and $\dddot{x} = n \times \overline{n-1} \times \overline{n-2}\dot{y}^3y^{n-3}$.

From whence, if the index n is any whole poſitive number, x will have as many ſubordinate fluxions as are units in n, and no more. For when $n = 1$; then $\dot{x} = \dot{y}$ and $\ddot{x} = o$; when $n = 2$, $\dot{x} = 2\dot{y}y$, $\ddot{x} = 2\dot{y}^2$ and $\dddot{x} = o$, and when $n = 3$; $\dot{x} = 3\dot{y}y^2$; $\ddot{x} = 6\dot{y}^2y$ and $\dddot{x} = 6\dot{y}^3$.

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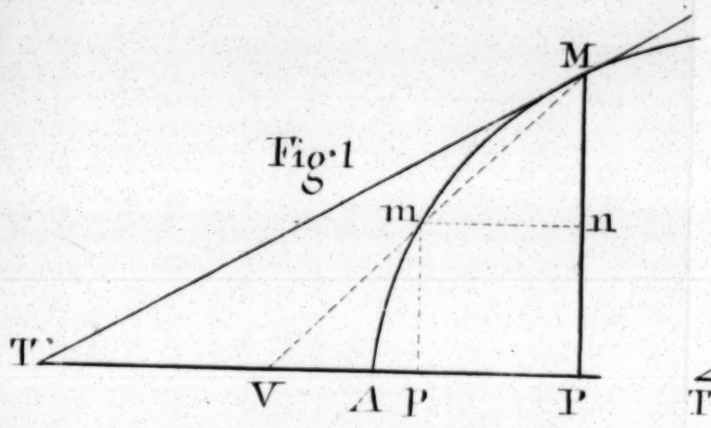
Plate 1.

Fig. 1



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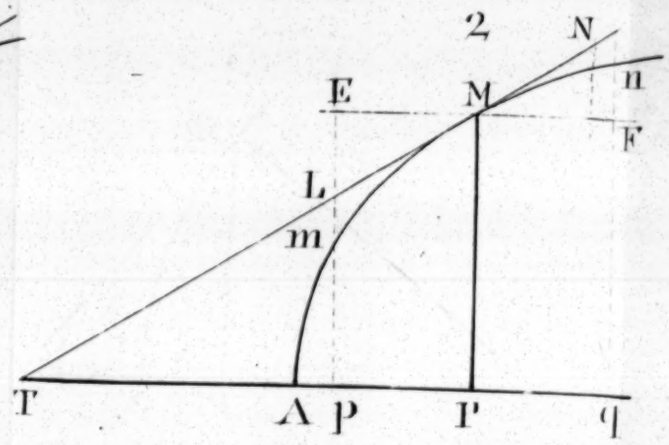

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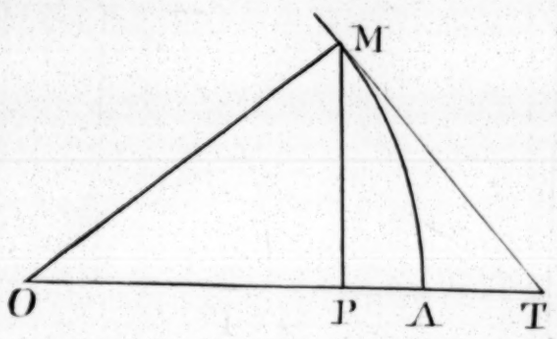

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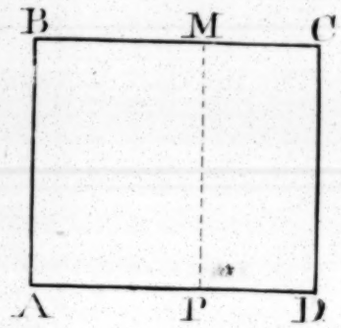

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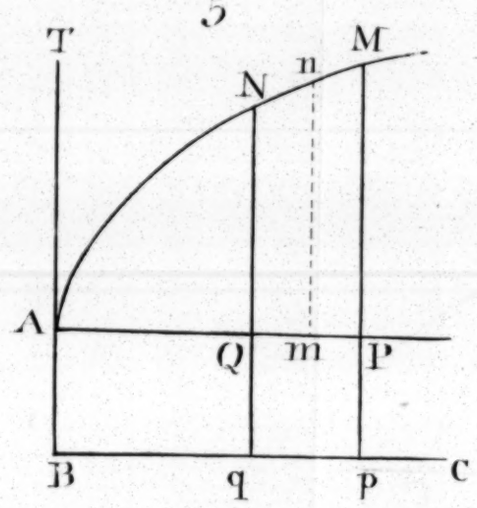

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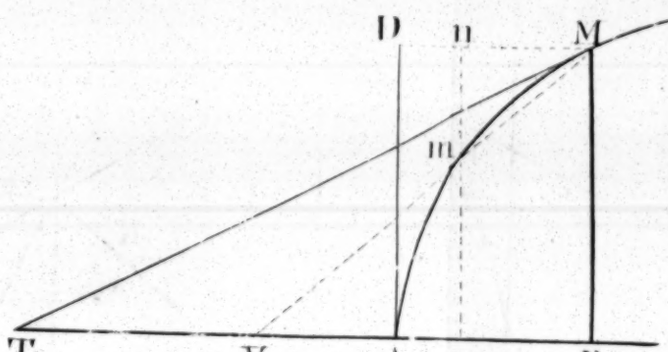

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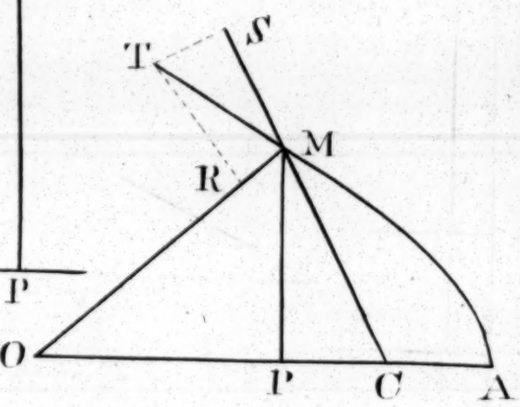

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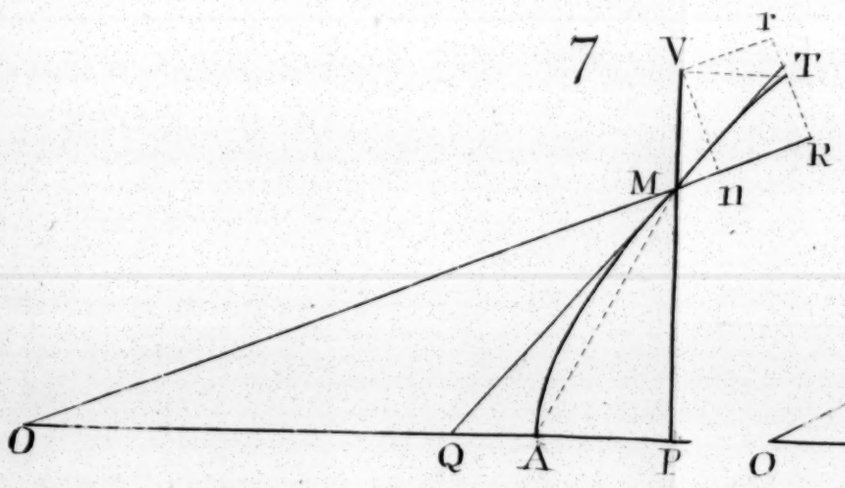

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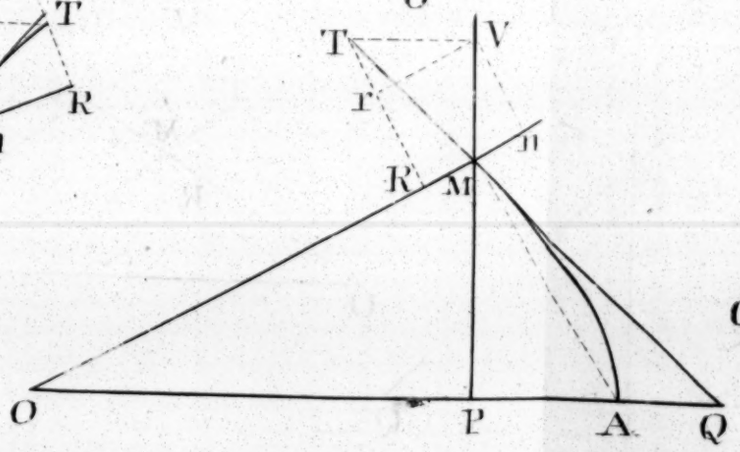

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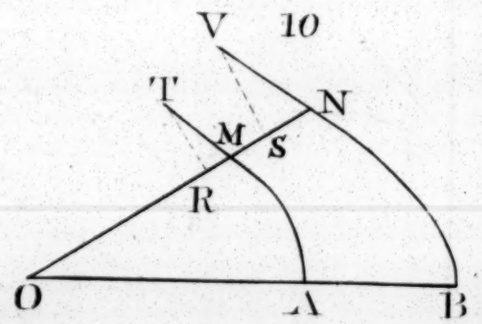
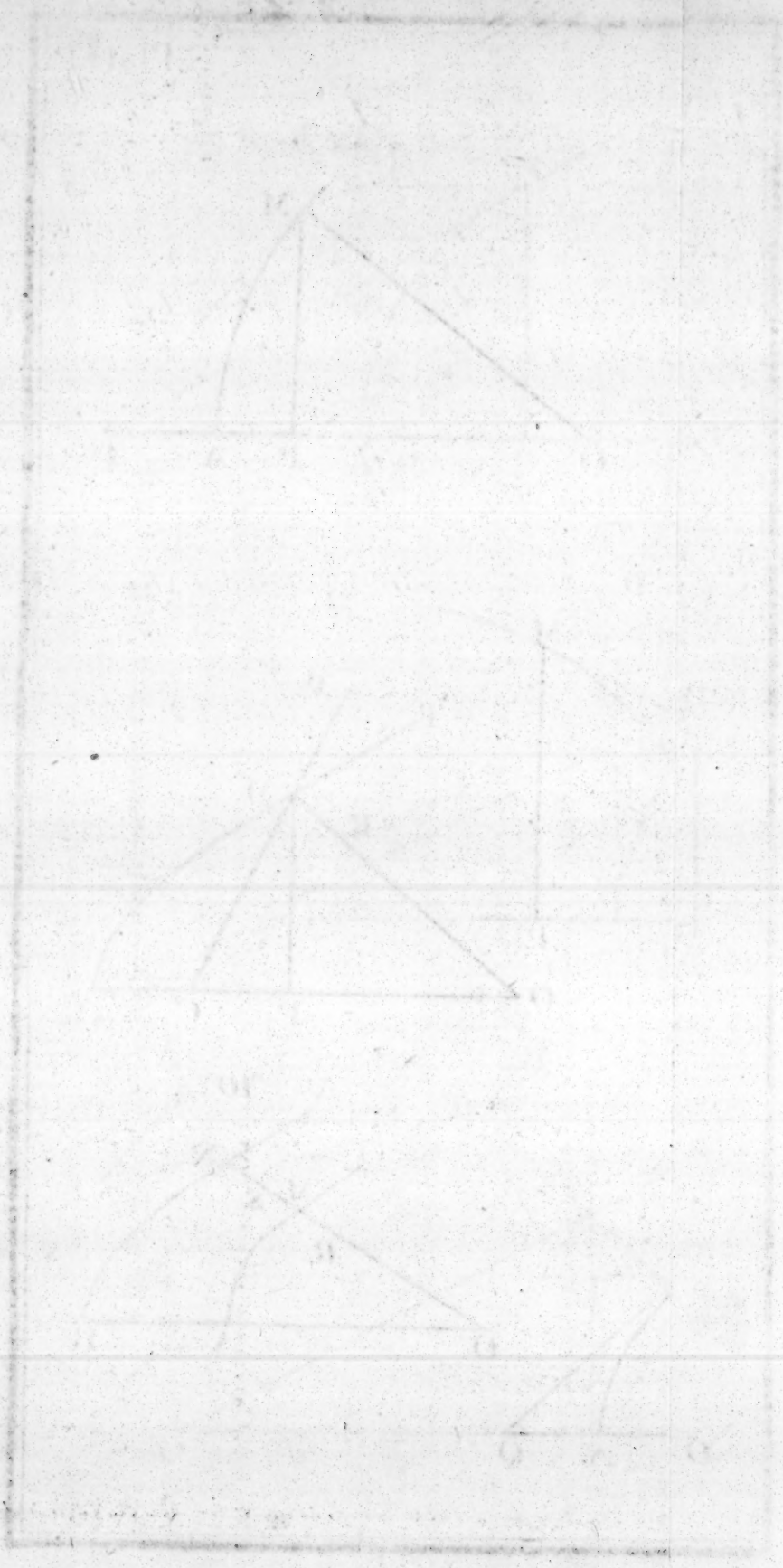




SECTION II.

The METHOD of finding the greateſt and leaſt values of variable QUANTITIES or MAGNITUDES.

PROBLEM. *Plate* II. *Fig.* 11, 12, 13, 14.

26. TO *find the greateſt and leaſt values of a variable quantity or magnitude.*

Let the ordinate PM of a curve ME, be expreſſed by powers of the abſciſſa AP and conſtant quantities: it is evident, that when that ordinate becomes either the greateſt or leaſt DE; the ſubtangent TP, becomes infinite or nothing in that caſe, ſince the tangent TM becomes parallel to the axis AP, in the firſt caſe, and coincides with the ordinate DE, in the ſecond; the fluxion of the ordinate becomes therefore o in the firſt caſe and that of the abſciſſa in the ſecond. From whence follows this general

RULE.

Make the fluxion of the expreſſion or ordinate equal to 0; *then the value found of the abſciſſa or variable quantity will determine the value ſought.*

As the value of the variable quantity thus found, may give either the greateſt or the leaſt quantity, it is neceſſary to obſerve, that if any other value of the variable quantity be taken, and the quantity be always leſs than that found, it will be the greateſt, or if they are always greater, the leaſt of all.

EXAMPLE.

27. Let $yy = 2ax - xx$, be the equation of the curve, AP $= x$ and PM $= y$; then $o = 2a\dot{x} - 2x\dot{x}$ or $x = a$; this value wrote into the equation gives $yy = aa$ or $y = a$, and y is the greateſt ordinate: for when x is either o or $= 2a$, y becomes o.

N. B. Obſerve that when y is either the greateſt or the leaſt, any of its powers, or the ſum of the pro-

ducts of y and conſtant quantities, are likewiſe the greateſt or leaſt, as may be eaſily proved.

PROBLEM.

28. *To divide a given line a into two ſuch parts* x, y, *that* $x^m y^n$ *ſhall be the greateſt of all the like products.*

The fluxion $m\dot{x}y^n x^{m-1} + n\dot{y}x^m y^{n-1}$ divided by $x^m y^n$, and made equal to o gives $\frac{m\dot{x}}{x} + \frac{n\dot{y}}{y} = o$, or $my\dot{x} + nx\dot{y} = o$: but $a = x + y$, by ſuppoſition, and therefore $\dot{x} + \dot{y} = o$, or $-\dot{x} = \dot{y}$; hence $my\dot{x} + nx\dot{y} = o$, gives $my = nx$. Which ſhews, that if the given line be divided into two parts proportional to the indices m, n, the product $x^m y^n$, will be the greateſt of all.

For ſince $y = a - x$, the product $x^m y^n$ becomes $x^m \times \overline{a-x}^n$; now when x is either o or $= a$, that expreſſion vaniſhes or is $= o$; the value found is therefore the greateſt.

Becauſe $my = nx$ and $y = a - x$, we get $ma - mx = nx$, or $\frac{ma}{m+n} = x$: now if n be negative and leſs than m; then $\frac{ma}{m-n} = x$, and ſince m is greater than $m - n$, x muſt be greater than a; and this cannot be unleſs we ſuppoſe $y = a + x$, which makes $x^m y^n$, or its equal $x^m \times \overline{x-a}^{-n}$, to be the leaſt of all, when $my = nx$, becauſe if $x = a$, or x infinite, the expreſſion is infinite.

PROBLEM.

29. *To divide a given line a into three parts* x, y, z, *ſuch that the product* $x^m y^n z^r$ *ſhall be the greateſt of all.*

It is evident, that when z is conſtant, and $x^m y^n$ the greateſt, the product $x^m y^n z^r$ will alſo be the greateſt; and when x is conſtant, and $y^n z^r$ the greateſt, the product $x^m y^n z^r$, will be likewiſe the greateſt; but when x is conſtant we get $ry = nz$, by the laſt, and when z is conſtant $my = nx$. From theſe two equalities we get $m : x :: n :$

$x::n:y::r:z$; which ſhews that if the line a be divided into three parts proportional to the indices m, n, r; the product above will be the greateſt of all. For $y=a-x-z$, by ſuppoſition, the product $x^m y^n z^r$, or its equal $x^m z^r \times \overline{a-x-z}^n$ becomes $=o$, when x is either o or $=a-z$; this product is conſequently the greateſt of all, when $ry=nz$ and $my=nx$.

It may be proved in the ſame manner, that if a line be divided into ever ſo many parts proportional to the indices of the powers, then the product of theſe powers will be the greateſt of all. Thus the ſquare of half a line is greater than any rectangle that can be made by any two unequal parts, and the cube of one third of a line greater than the parallelepiped of any three other parts.

N. B. When the variable quantity contained in the equation found by making the fluxion of the expreſſion equal to o, has ſeveral poſitive values, the queſtion contains more than one greateſt or leaſt, and they are alternately the greateſt and leaſt: that is, if the firſt is a greateſt, the ſecond will be the leaſt, and the third the greateſt, and ſo on; or if the firſt is the leaſt, the ſecond will be the greateſt: but the leaſt between the two greateſt may ſometimes be o, and the greateſt between two leaſt infinite.

THEOREM. *Fig.* 15.

30. *Of all the triangles which have equal baſes and equal angles at the vertex, the iſoceles contains the greateſt area.*

Upon the given baſe AC as a chord, deſcribe the arc ABC of a circle, ſo as to contain the given angle: then becauſe the line BE, which biſects the chord AC at rightangles, paſſes thro' the center of the circle, and is therefore the greateſt that can be drawn from any point in the arc ABC, by the property of the circle; and triangles which have equal baſes are as their altitudes; the iſoceles triangle ABC is greater than any other triangle of the ſame baſe, and the ſame angle at the vertex.

 31. Hence,

31. Hence, of all rectilineal figures, terminated by the ſame number of equal ſides, that which is inſcribed in a circle contains the greateſt area. For if the radii are drawn to all the angles, it will be divided into as many iſoceles triangles as there are ſides, and each of which being greater than any other of the ſame baſe and angle at the vertex; their ſum or the whole figure inſcribed in a circle will conſequently be greater than any other of the ſame number of equal ſides.

COR.

32. It is manifeſt, that of all figures whatever, which have the ſame boundary, the circle is the greateſt, and of all equal figures, the circle has the leaſt boundary. For ſince circles are always greater than any inſcribed figure, and the inſcribed figures being greater than any other of the ſame boundary; the circle will be greater than any other figure of the ſame boundary, or of all equal figures has the leaſt boundary.

PROBLEM. *Fig.* 16.

33. *Let a ſquared timber* BD, *placed horizontally, be ſupported by a prop at one end* D, *and by a ſtrut* AB *on the other* B; *to find the poſition of the brace* AC, *of a given length, ſo as to ſupport the timber* BD *in the beſt manner poſſible.*

Draw AF perpendicular to BD and BE to AC: then from a known principle in mechanics, if AC expreſſes the force with which the timber BD is ſupported in a perpendicular direction, AF will expreſs the force by which it is ſupported in this oblique direction; therefore the rectangle made by that force and the diſtance BC of the point C from the point fixed B, expreſſes the momentum of that force, which ſhould be the greateſt poſſible. Now the rightangled ſimilar triangles AFC, BEC, give AC : AF :: BC : BE, or AF × BC = AC × BE: whence the momentum of this force is proportional to the area of the triangle ABC,

ABC, whoſe baſe is given as well as the angle B at the vertex; which is the greateſt, when AB=BC, by *Art.* 30.

COR. *Fig.* 17.

34. If the ſquared timber BD be fixed at one end B, and ſupported by a prop AC, of a given length, it is plain that when AC makes equal angles with the timber AD, and the line BA, it will ſupport the timber AD in the beſt manner. For if BE be perpendicular to AC, the product BE×AC, expreſſes the momentum of the force with which the prop ſupports the timber, which is the greateſt poſſible when BA=BC, by *Art.* 30.

COR.

35. Since the length of the prop AC, has been taken at pleaſure, in the two laſt articles, it may have one more advantageous than any other, which is, when it ſupports the timber in the middle, ſuppoſing the timber fixed at both ends.

PROBLEM. *Fig.* 18.

36. *To find the beſt angle of a roof in reſpect to the ſtrength of the principal rafters* AD, BD.

Let DC be perpendicular to AB, and AE to BD produced: then as the ſtrength of the rafters decreaſes as their length increaſes, and their weights increaſes as their length increaſes, their ſtrength decreaſes likewiſe on that account; that is, the ſtrength of rafters decreaſes in a duplicate ratio, to that of their length. Now as the abſolute force of the rafter AD, is to that part by which it acts in the oblique direction, as AD is to AE, the momentum of the ſtrength of the rafter AD is therefore as AE directly, and the ſquare of AD or BD inverſely.

By the ſimilarity of the triangles AEB, DCB, we have BD : BC :: BA : BE; and ſince BC as well BA, are given, BE is as BD inverſely: whence AE×BE^2, muſt be the greateſt poſſible. If AB=*a*, AE=*x*; then

will $BE^2 = aa - xx$, and $aax - x^3$, the greateſt, whoſe fluxion made $= o$, gives $aa = 3xx$. For when $x = a$ or $= o$, the expreſſion becomes o. By the ſimilarity of triangles we get $CD^2 = \frac{1}{3} aa$, and by extracting the ſquare root $CD = \frac{6}{17} a$ nearly. This anſwers nearly the practical rule uſed by carpenters, of making the height CD one-third of the baſe AB.

PROBLEM. *Fig.* 19.

37. *To find a point* D, *within a given triangle, ſuch that the ſum of the lines drawn from this point to the three angles, may be the leaſt poſſible.*

Suppoſe one of theſe lines AD, as invariable, from the center A with the radius AD deſcribe an arc of a circle, and from any point a in the tangent at D, draw a b perpendicular to BD and a c to CD produced: then will Db, be the fluxion of DB, and Dc, that of CD, by *Art.* 24: and ſince the ſum of theſe lines is given their fluxions muſt be equal, and as Da is common to the right-angled triangles Dca, Dba, they are equal in all reſpects: whence by adding the angle aDb to the right angle ADa, we get the angle ADB, and by adding aDc or its equal dDC to the right angle ADd, we get the angle ADC equal to ADB. It may be proved in the ſame manner, by ſuppoſing CD conſtant, that the angles CDA, CDB, are equal conſequently, when the three angles at the point D, are equal, the ſum of the three lines AD, BD, CD, will be the leaſt poſſible.

PROBLEM. *Fig.* 20.

38. *To find a point within a quadrilateral* ABCD, *ſo that the ſum of the four lines drawn from it to the angles, ſhall be the leaſt poſſible.*

The interſection E of the two diagonals AC, BD will be the point required. For if from any other point F, lines are drawn to the oppoſite angles A, C, the

the ſum of the lines AF, CF, will always be greater than the diagonal AC: and for the ſame reaſon, the ſum of the lines drawn from the point F to the oppoſite angles B, D, will always be greater than the diagonal BD, the interſection E of the diagonals is therefore the point required.

What has been ſaid on this ſubject in our elements of mathematics, and here together, with what will be ſaid more hereafter, will ſufficiently explain this very uſeful method, in all mechanical ſubjects.

SECTION III.

Of the RADII of CURVATURE.

WHEN a circle touches a curve ſo as to have the ſame fluxion in that point, the radius of this circle is called the *Radius* of *Curvature* in that point.

PROBLEM. *Fig.* 21.

39. *To find the radius of curvature* DM, *by means of the lines drawn from the given point* C.

From any point T in the tangent at M, and the extremity D of the radius of curvature, draw TR and DE, perpendicular to CM, and from the point C, the line CS to MT: let DM$=r$, CM$=y$, CS$=s$, and MT$=\dot{z}$, MR$=\dot{y}$, TR$=\dot{x}$; then the angular motion of the ſides DM, MS, of the right angle DMS, at M, being always equal; the radius DM will be to the radius MS, as the fluxion $\dot{z}$ of the circular arc deſcribed by the point M is the fluxion $\dot{s}$ of the circular arc deſcribed by the point S in the ſame time; that is DM : MS :: $\dot{z}$: $\dot{s}$; and the ſimilar triangles TRM, CSM, give MS : CM :: MR($\dot{y}$) : TM($\dot{z}$); the compound of theſe two proportions give $r : y :: \dot{y} : \dot{s}$, or $r\dot{s}=y\dot{y}$. But CM(y) : CS(s) :: DM(r) : ME$=\frac{rs}{y}$. By writing the value of r, in $r\dot{s}=y\dot{y}$, into that of ME, we get $\dot{s}\times$ME$=s\dot{y}$

EXAMPLE. *Fig.* 22.

40. Let AM be a common parabola, A its vertex, and C the focus; then by the property of that curve, the tangent at M meets the tangent at A, in the ſame point S, as the perpendicular drawn from the focus to the tangent MS; the right angled triangles CAS, CSM, are therefore ſimilar; that is, if $CA=a$, we get $a:s::s:y$, or $ay=ss$, whoſe fluxion is $a\dot{y}=2s\dot{s}$, and as $r\dot{s}=y\dot{y}$, we get $ar=2sy$.

CONSTRUCTION.

Draw ME parallel to the axis AC, and equal to twice CM; then the perpendicular ED to ME determines the radius required. For the ſimilar triangles SAC, DEM, give AC (a) : CS (s) :: ME ($2y$) : MD (r) or $ar=2sy$.

When the point M coincides with the vertex A of the axis, it is plain that CM, CS become equal to CA, that is $s=y=a$, ſo then $ar=2sy$, gives $r=2a$, which ſhews that the radius of curvature is equal to half the parameter of the axis at the vertex.

EXAMPLE. *Fig.* 23.

41. Let AM be an ellipſis or hyperbola, C the center and Nn the conjugate to the diameter Mm: if $CN=u$, a, b, half the firſt and ſecond axis, then will $us=\mp ab$, by the properties of theſe curves, whoſe fluxion is $-s\dot{u}=u\dot{s}$; and ſince the ſum or difference between the ſquares of any two conjugate diameters is always conſtant, that is $yy\pm uu$ is conſtant, we $\mp y\dot{y}=u\dot{u}$, the compound of this and $-s\dot{u}=u\dot{s}$ gives $\mp sy\dot{y}=uu\dot{s}$; this and $\dot{s}\times ME=s\dot{y}$, by *Art.* 39. gives $y\times ME=\pm uu$.

CONSTRUCTION.

In CM take ME towards the center in the ellipſis or from the center in the hyperbola, ſo that $\div CM : CN : ME$, then ED perpendicular to ME, will determine the radius required.

When

When the point M coincides with the vertex A or B, the radius of curvature becomes equal to the parameter of half the firſt or ſecond axis.

OTHERWISE.

42. Becauſe $r\dot{s}=y\dot{y}$, by *Art.* 39, and $us=ab$, by the property of the curve, or $s=\frac{ab}{u}$, whoſe fluxion is $-\dot{s}=\frac{ab\dot{u}}{uu}$, this and $r\dot{s}=y\dot{y}$ gives $-\frac{rab\dot{u}}{uu}=y\dot{y}$; but $y\dot{y}\pm u\dot{u}=0$, therefore $rab=\pm u^3$, by ſubſtitution. Now if MD interſects the firſt axis AC, in K and MK$=c$; then will $ac=bu$, by the nature of the curve. Hence $rab=\pm u^3$ becomes $rb^4=aac^3$, or if $ap=bb$, we get $rpp=c^3$. Conſequently the radius MD, is a fourth continued proportional to the parameter of half the firſt axis AC, and to the perpendicular MK.

No. 2. Since (*Fig.* 21) TM ($\dot{z}$) : TR ($\dot{x}$) :: CM (y) : CS (s) or $s\dot{z}=y\dot{x}$, and if we ſuppoſe the point C to remove at an infinite diſtance from the point M, ſo that the ordinates CM may become parallel and equal, then y being infinite, may be conſidered as conſtant the the fluxion of $s\dot{z}=y\dot{x}$, compared with $r\dot{s}=y\dot{y}$, gives $r\ddot{x}=\dot{y}\dot{z}$, or $-\dot{y}\dot{z}^2=r\dot{x}\ddot{z}$ according as $\dot{z}$ or $\dot{x}$ is conſtant, for the equations, when the ordinates are parallel.

There are different ways of finding the radii of curvatures, but none more eaſy and ſo ſhort as this, nor that leads more directly to the application; the many examples of curves that may be deſcribed, and the finding the nature of the curve which is continually touched by the radii of curvature, are no more than a uſeleſs ſpeculation; we ſhall therefore take no notice of them here, excepting the following one uſeful in the theory of pendulums.

DEFINITION. *Fig.* 24. *Plate* III.

If a ſemi-circle ANB, rolls upon a right line BD, the

the curve AMD deſcribed by the point A in the circumference, is called a *Cycloid.*

COR.

43. Hence, when the deſcribing point A, arrives to any other point M; the chord ML, drawn from the point M to the point of contact L, will from the nature of the deſcription be perpendicular to the tangent TM of the curve at M, and this tangent will be the chord of the complement of LM to the ſemi-circle. It appears alſo that the baſe BD is equal to the ſemi-circumference ANB of the generating circle, the part BL to the arc TM, and the part LD to the arc ML.

PROBLEM.

44. *To find the length of any arc* AM *of the cycloid.*

Draw PM perpendicular to the diameter TL, which itſelf is perpendicular to the baſe BD, and call LT or AB, a, TP$=x$, the chord TM$=v$, and the arc AM $=z$: then becauſe TP : TM or TM (v) : LT (a) :: $\dot{x}$: $\dot{z}$, by *Art.* 4. we have $v\dot{z}=a\dot{x}$; and ſince LT (a) : TM (v) :: TM (v) : TP (x) or $ax=vv$, by the ſimilarity of triangles, whoſe fluxion is $a\dot{x}=2v\dot{v}$, whence ($a\dot{x}=$) $v\dot{z}=2v\dot{v}$, or $\dot{z}=2\dot{v}$: and as the fluxion $\dot{z}$ of the arc is always double the fluxion $\dot{v}$ of the chord TM; the arc AM will likewiſe be double the chord TM, by *Art.* 7. the whole curve AMD is therefore double the diameter AB of the generating circle.

PROBLEM.

45. *To find the radius of curvature* CM *at any given point in the cycloid.*

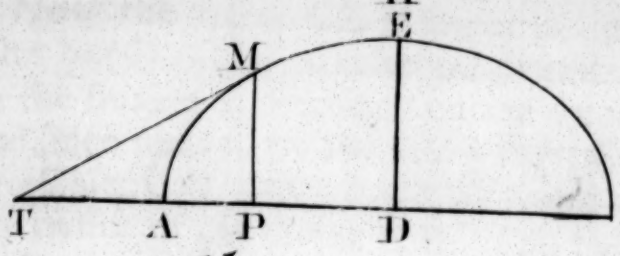
In the baſe BD, take bL$=\dot{y}$ for the fluxion of BL or its equal the arc TM, and draw ba perpendicular to CM: then the ſimilar triangles TLM, bLa, give TL : LM :: bL ($\dot{y}$) : ba: but the radius TL is to the ſine LM of the angle LTM, as the fluxion $\dot{y}$ of the arc is to the fluxion $\dot{v}$ of the coſine TM, by *Art.* 4.

No. 2. Hence $\dot{y}$: ba :: $\dot{y}$: $\dot{v}$ = ba, by equality of ratios.

Now

Plate II.

11



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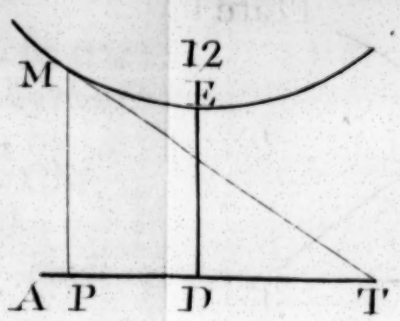

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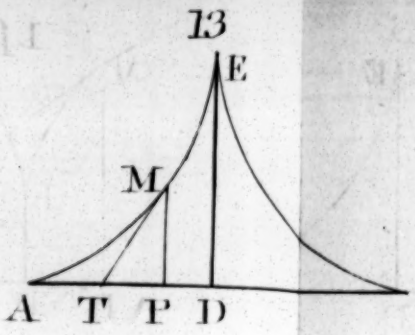

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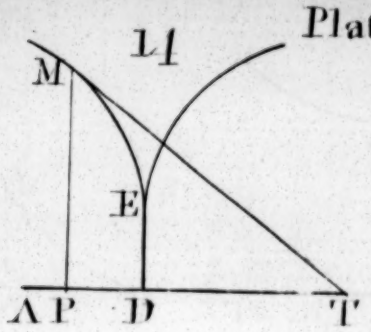

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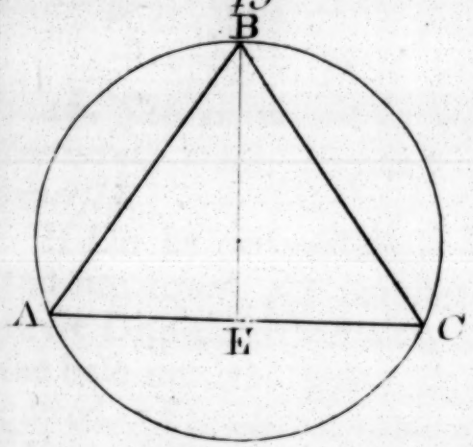

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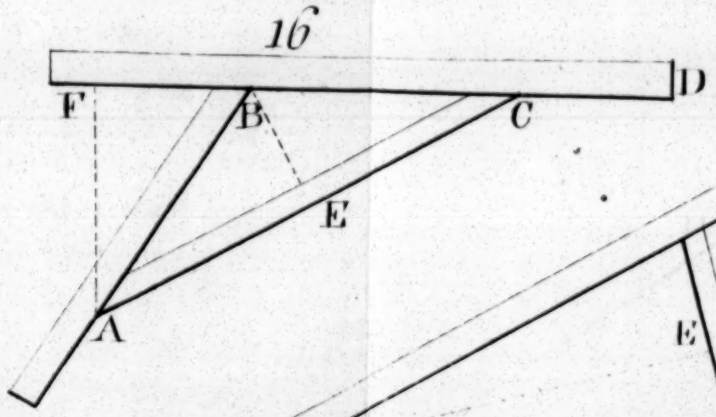

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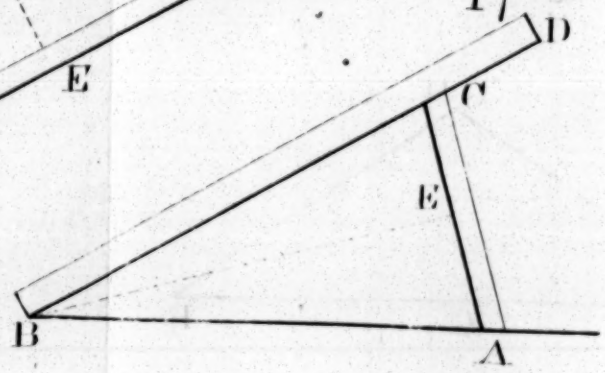

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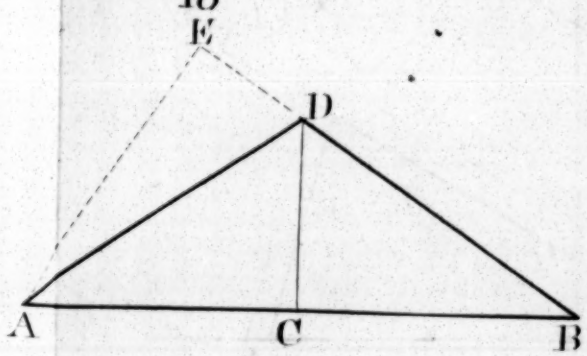

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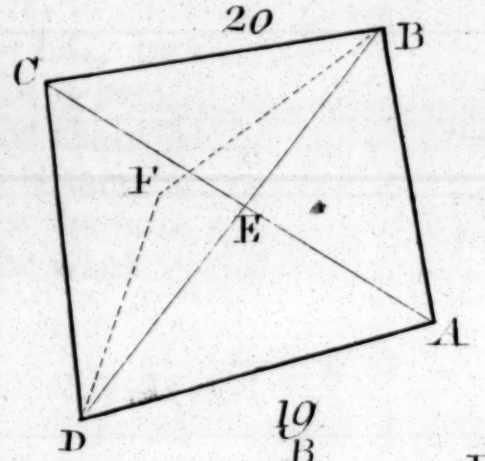

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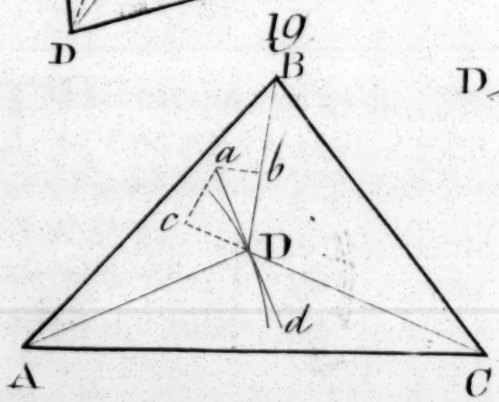

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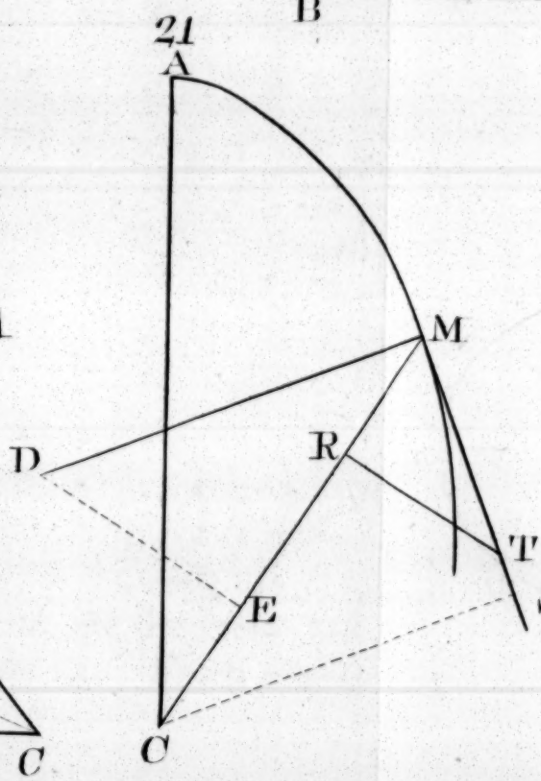

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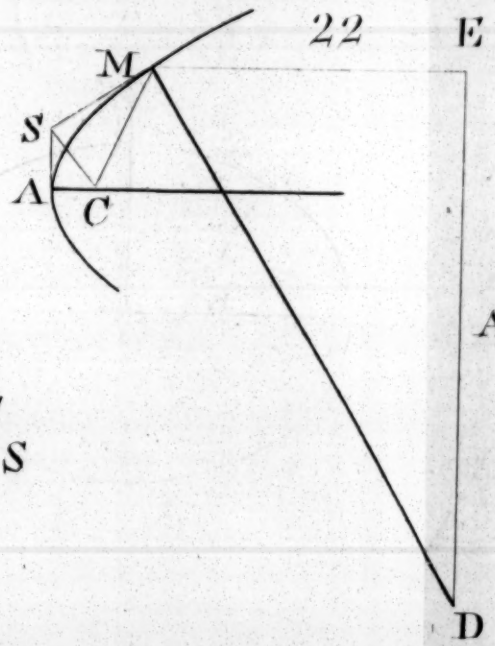

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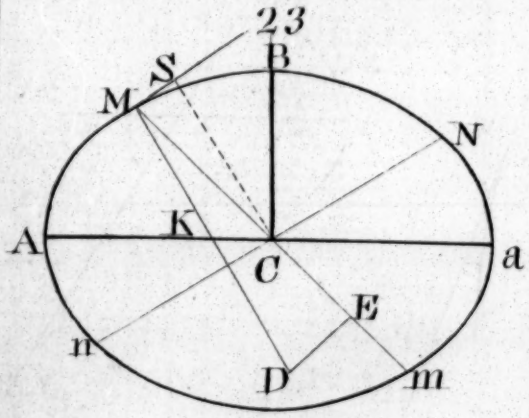
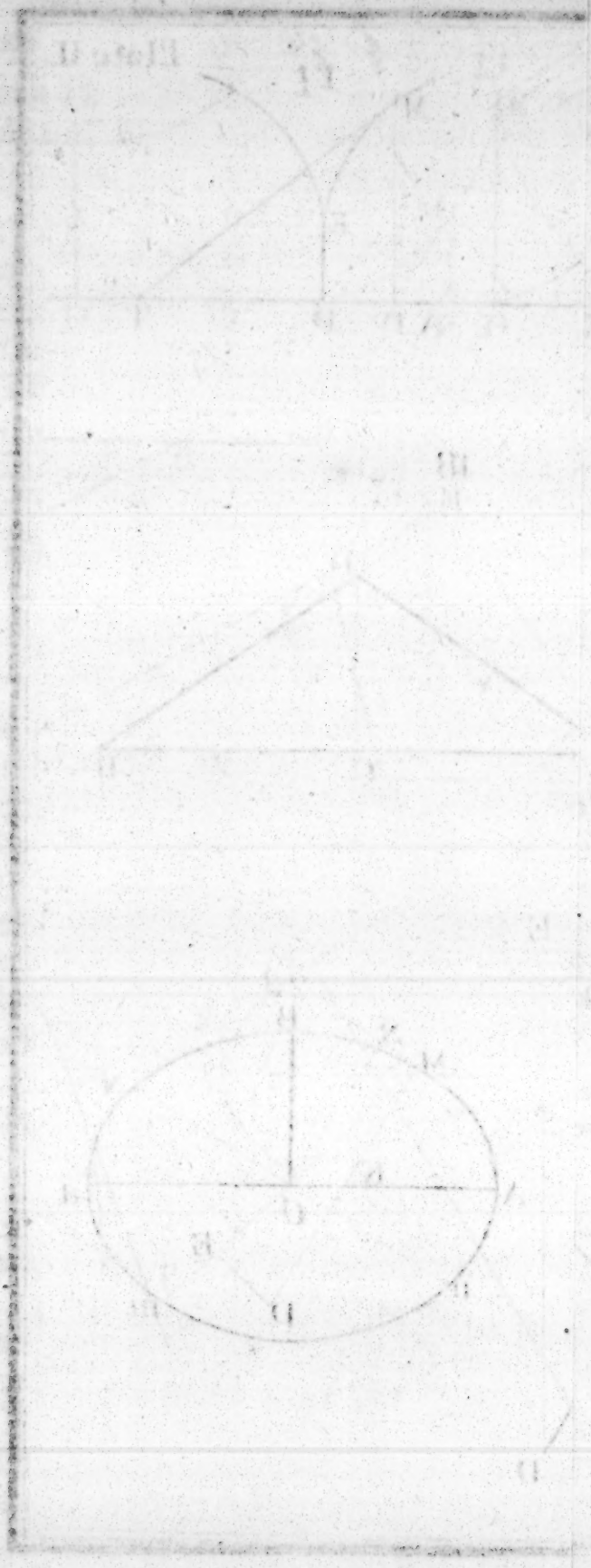




Now the radius CL is to the radius CM as the fluxion ba ($\dot{v}$) of the circular arc deſcribed by the firſt, is to the fluxion $\dot{z}$ deſcribed by the ſecond, by *Art.* 25. and ſince $\dot{z}=2\dot{v}$, by the laſt we get CL : CM :: $\dot{v}$: $2\dot{v}$, or 2CL=CM, and CL=LM; and at the vertex A, the radius of curvature is equal to twice the diameter of the generating circle.

THEOREM.

46. *The curve* mCD *which the radii of curvature continually touch, is another cycloid equal to the firſt, but reverſed.*

If Dn be perpendicular to BD, and equal to AB, and nm equal and parallel to DB, then if the ſemicircle Dkn deſcribed with the diameter nD, rolls upon the baſe nm, the point D will deſcribe the curve required. For if the chord Dk be drawn parallel to MC, and Ck joined; then the parallel chords ML, Dk, in equal circles are equal, and ſince ML is equal to LC, by the laſt, CD is a parallelogram, and ſo Ck is parallel and equal to LD. But MC is double the chord ML, or equal to the arc DC of the curve, it is therefore equal to the arc AM, by *Art.* 44; and for the ſame reaſon the arcs DM, mC are equal, and the whole cycloid AMD to the whole cycloid DCm.

SECTION IV.

Of FLUENTS.

THE method of finding fluents, is the inverſe of that of finding fluxions. For the fluent of $n\dot{y}^{n-1}$ is y^n, that of $ncr\dot{z}z^{n-1}\times\overline{a+cz^n}^{r-1}$, is $\overline{a+cz^n}^r$; and that of $\dot{y}z+y\dot{z}$ is yz; for theſe fluents are the very ſame as thoſe propoſed in *Art.* 9, 10, 14. From whence we deduce the following

RULE I.

47. *To find the fluent of a ſingle quantity* $\dot{y}y^n$, *raiſed to any power.*

Change

Change the fluxion $\dot{y}$ into a fluent, and divide by the index $n+1$ thus increaſed.

RULE II.

48. *To find the fluent of a binomial* $\dot{z}z^{n-1} \times \overline{a+cz^n}^r$, *whoſe fluxion is in a conſtant ratio to that which multiplies the binomial*

Increaſe the index r of the binomial by unity, and divide by the product of the index thus increaſed and the fluxion of the binomial.

EXAMPLES.

49. The fluent of $a\dot{x}+b\dot{x}x+c\dot{x}x^2+d\dot{z}z^3$, is $ax+\frac{1}{2}bx^2+\frac{1}{3}cx^3+\frac{1}{4}dz^4$, by the firſt rule.

The fluent of $x\dot{x} \times \overline{aa+xx}^{\frac{1}{2}}$, is found by adding unity to the index $\frac{1}{2}$, which gives $\frac{1}{2}+1$, or $\frac{3}{2}$, this multiplied by the fluxion $2x\dot{x}$ of the binomial and divided by the product $3x\dot{x}$, gives $\frac{1}{3} \times \overline{aa+xx}^{\frac{3}{2}}$, for the fluent required by rule the ſecond.

If $x\dot{x} \times \overline{aa+xx}^{-\frac{1}{2}}$; then unity added to the index $-\frac{1}{2}$ gives $1-\frac{1}{2}$ or $\frac{1}{2}$, this multiplied by $2x\dot{x}$ the fluxion of the binomial $aa+xx$, and the fluxion $x\dot{x} \times \overline{aa+xx}^{\frac{1}{2}}$ divided by the product $x\dot{x}$ gives $\overline{aa+xx}^{\frac{1}{2}}$, or $\sqrt{aa+xx}$ for the fluent required.

N. B. When a fluent has a conſtant term, it is generally incompleat, becauſe the conſtant terms vaniſh in the fluxion, but may be compleated; for if the fluent ſhould become o, when the variable quantity is of a certain value, what remains in this caſe, muſt be added with a contrary ſign. Thus if $\sqrt{aa-xx}$, ſhould become o, when $x=o$; then becauſe when $x=o$, there remains $\sqrt{aa}$ or a: we get $\sqrt{aa-xx}-a$ for the compleat fluent. But if $\sqrt{aa-xx}$ ſhould vaniſh when $x=a$, it requires no addition.

PRO-

PROBLEM. *Fig.* 25.

50. *To expreſs the circular arc* AM *by the radius* CA (*r*) *and the tangent* AT (*z*).

The ſquare $rr+zz$ of CT, being to the ſquare rr of the radius CM, by *Art.* 25, as the fluxion $\frac{1}{2}r\dot{z}$ of the triangle CAT, to the fluxion $\frac{\frac{1}{2}\dot{z}r^3}{rr+zz}$ of the ſector CAM, which divided by $\frac{1}{2}r$ gives $\frac{r\dot{z}}{rr+zz}$, for the fluxion of the arc AM, by *Art.* 21. Now this fluxion reduced to an infinite ſeries by a continual diviſion, ſuppoſing $r=1$, gives $\dot{z}-\dot{z}z^2+\dot{z}z^4-\dot{z}z^6+$, &c. whoſe fluent, by the firſt rule, is $z-\frac{1}{3}z^3+\frac{1}{5}z^5-\frac{1}{7}z^7+\frac{1}{9}z^9-$, &c. Or if $A=z$, $B=zzA$, $C=zzB$, $D=zzC$, &c. This fluent becomes $A-\frac{1}{3}B+\frac{1}{5}C-\frac{1}{7}D+\frac{1}{9}E-$, &c.

As this ſeries converges but very ſlowly unleſs z is very ſmall, we ſhall give the following

LEMMA. *Fig.* 26.

51. *The tangents* $AT=a$, *and* $AB=b$, *of any two arcs* AF, AM, *being given, to find the tangent* $FG=t$, *of their difference* FM.

Draw GD perpendicular to CA, interſecting CB in in E, make $CA=1$, and $CB=x$: then the ſimilar triangles CAB, GFE, give $CA:AB::GF:EF=bt$, or, $CE=1-bt$, and $CA:CB::GF:GE=tx$, and by the parallels GD, TA, we have $CE:CB::GE:TB$, that is, $1-bt:x::tx:a-b$, or $txx=\overline{a-b}\times\overline{1-bt}$, or becauſe $xx=1+bb$, we get $t+bbt=a-b-abt+bbt$, and therefore $t=\frac{a-b}{1+ab}$, by reduction.

COR. *Fig.* 27.

52. Hence, if from any arc AE, another arc AB be taken as often as can be done, and the point F be neareſt to the point E; by calling $AE=A$, $AB=B$,

BE

BE=C, CE=D, DE=E, FE=F, and their reſpective tangents, a, b, c, d, e; and ſuppoſing $a=1$, $b=\frac{1}{5}$, the other tangents are as follows,

$$A-B=C,\ c=\tfrac{2}{3}.$$
$$C-B=D,\ d=\tfrac{7}{17}:$$
$$D-B=E,\ e=\tfrac{9}{46}.$$
$$B-E=F,\ f=\tfrac{1}{239}.$$

Now, if the three firſt equalities of the arcs are added together, then will $A-3B=E$, and this equality ſubtracted from the fourth, gives $4B-F=A$; that is, four times the arc B, whoſe tangent is $\frac{1}{5}$, leſs the arc F whoſe tangent is $\frac{1}{239}$, is equal to an arc of 45 degrees.

In the ſame manner other tangents may be found, ſo as their ſums or difference ſhall be equal to a given arc. Thus if $c=\frac{1}{7}$, $d=\frac{3}{79}$; then will $2D+5C=45$ degrees, or if $a=\frac{1}{12}\sqrt{3}$, and $b=\frac{1}{37}\sqrt{3}$, then $3A+2B=30$ degrees.

EXAMPLE.

53. Let $z=\frac{1}{239}$, then by reducing the ſeries $A-\frac{1}{3}B+\frac{1}{5}C-\frac{1}{7}D+\frac{1}{9}$, &c. into decimals, and ſetting all the terms of the ſame ſign under each other, we get,

$$+0.00418,41004,18410,04184,101=A.$$
$$256,47231,442=\tfrac{1}{5}C:$$
$$4=\tfrac{1}{9}D.$$

$$+0.00418,41004,18665,51415,574.$$

$$-0.00000,00244,16591,78708,380=\tfrac{1}{3}B.$$
$$320,713=\tfrac{1}{7}D:$$

$$-0.00000,00244,16591,79029,093.$$

Hence we get 0.00418,40760,02074,72386,454 = the arc F.

And

And if we ſuppoſe $z=\frac{1}{5}=.2$; the ſeries above gives

$+0.20006,40568,88888,88888,889=A+\frac{1}{5}C.$
$63015,38461,538=\frac{1}{9}E.$
$77,10117,647=\frac{1}{13}G.$
$9986,438=\frac{1}{17}I.$
$13,422=\frac{1}{21}L.$
$18=\frac{1}{25}N.$

$+0.20006,40569,51981,47467,952.$

$-0.00266,66666,66666,66666,666=\frac{1}{3}B.$
$18285,71428,57142,857=\frac{1}{7}D.$
$18,61816,18181,818=\frac{1}{11}F.$
$2184,33333,333=\frac{1}{15}H.$
$2,75941,053=\frac{1}{19}K.$
$364,722=\frac{1}{23}M.$
$498=\frac{1}{27}O.$

$-0.00266,84971,02100,71630,947.$

Hence 0.19739,55598,49880,75837,005=to the arc B.

Now becauſe four times the arc B, leſs the arc F, is equal to 45 degrees, 16 times the arc B leſs 4 times the arc F, will be equal to half the circumference, which therefore is 3.14159,26535,89793,23846,264, true in all its places.

C O R.

54. Since unity is to 3.14159, &c. as the radius a of any circle is to $a\times 3.14159$ the ſemi-circumference of that radius; and if D expreſſes any number of degrees of that circle; then 180 degrees will be to D degrees as the ſemi-circumference $a\times 3.14159$ is to $.01745329\times aD$, that arc expreſſed by parts of the radius. Conſequently the number of degrees being given, the arc itſelf is given.

C O R.

55. If the diameter of a circle be 7, to find its circumference, ſay, 1 is to 3.14159 as 7 is to 21.99, or

22 nearly, which is Archimedes's proportion; or if the diameter be 113, then its circumference will be 354.99967 or 355 exceeding near; which is that of Snellius.

PROBLEM. *Fig.* 28.

56. *To find the area of an equilateral hyperbolic ſpace* BCMP, *terminated by two perpendiculars to the aſſymptote* AP.

Let AB=1, BC=a, and BP=x, then will PM=$\frac{a}{1+x}$, by the nature of the curve, and $\frac{a\dot{x}}{1+x}$ the fluxion of the ſpace BM, by *Art.* 16. which being reduced into an infinite ſeries by a continual diviſion, and the fluent found by *Art.* 47, gives $a\times$ by $x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+\frac{1}{5}x^5-$, &c.

But if Bp=x, then will pm=$\frac{a}{1-x}$, and $\frac{-\dot{x}a}{1-x}$ the fluxion of the ſpace pC, whoſe fluent being found as before is $-a\times$ by $x+\frac{1}{2}x^2+\frac{1}{3}x^3+\frac{1}{4}x^4+\frac{1}{5}x^5+$, &c.

COR.

57. Since $\frac{\dot{x}a}{1+x}$ is the fluxion of the hyperbolic ſpace BM or of the logarithm of AP, and $\frac{-\dot{x}a}{1-x}$ that of Ap, by the nature of the curve; it is manifeſt, that the fluxion of the logarithm of any number, is equal to the fluxion of that number divided by the ſame number.

COR.

58. From the nature of logarithms, the difference between the logarithms $a\times$ by $x-\frac{1}{2}x^2+\frac{1}{3}x-$, &c. and $-a\times$ by $x+\frac{1}{2}x^2+\frac{1}{3}x^3+$, &c. which is, $2a\times$ by $x+\frac{1}{3}x^3+\frac{1}{5}x^5+\frac{1}{7}x^7+$, &c. will be that of their quotient $\frac{1+x}{1-x}$. Or if A=1, B=xxA, C=xxB, D=xxC, and ſo on, this difference will be $2ax\times$ by $A+\frac{1}{3}B+\frac{1}{5}C+\frac{1}{7}D+$, &c.

EX-

EXAMPLE.

Let $\frac{1+x}{1-x}=2$, then will $x=\frac{1}{3}$, and this value of x, being wrote into the laſt ſeries, when reduced into decimals, gives

1.00000,00000
3703,70370
246,91358
19,59632
1,69351
15396
1447
139
15

1.03972,07708

This multiplied by $2ax$ or $\frac{2a}{3}$, gives $a \times .69314,71805$, for the logarithm of 2.

EXAMPLE.

Let $\frac{1+x}{1-x}=\frac{5}{4}$; then will $x=\frac{1}{9}$, and this value wrote into the ſeries gives

1.00000,00000
411,52263
3,04832
2688
26

1.00414,59809

This multiplied by $2ax$ or $\frac{2a}{9}$, gives $a \times .22314,35513$ for the logarithm of $\frac{5}{4}$. And as this number multiplied by 4 gives 5, by adding twice the logarithm of 2, we get $a \times$ by 1.60943,79124, for the logarithm

garithm of 5: and if the logarithm of 2 be added to that of 5, we ſhall have $a \times$ by 2.30258,50929 for the logarithm of 10.

C O R.

59. It is plain, that there may be as many ſorts of logarithms as there are different values taken for a; when $a=1$, they are called hyperbolic. When any particular ſyſtem of logarithms is given, the value of a may be found; thus in the common tables the logarithm of 10 is unity of 100 or 1000; its is 2 or 3: we have then no more to do than to make the logarithm of 10 found above equal to unity, or to divide unity by 2.3025, &c. which gives .43429,4482 for the value of a in that ſyſtem.

P R O B L E M. *Fig.* 29.

60. *To expreſs the area of the hyperbolic ſector* CAM, *by parts of the tangent* At.

Let CN be an aſymptote, and CM interſect the tangent AT at A in t. If $CA=a$, $AT=b$, and $At=z$; then the ſquare of Ct is to the ſquare of CM as the fluxion $\frac{1}{2}a\dot{z}$ of the triangle CAt is to the fluxion of the ſector CAM, by *Art.* 25; but, by the nature of the hyperbola, the ſquare of Ct is to the ſquare of CM, as the difference $bb-zz$, between the ſquares of the tangents AT and At, to the ſquare bb of AT: therefore $bb-zz : bb :: \frac{1}{2}a\dot{z} : \frac{abb\dot{z}}{2bb-2zz}=$ to the fluxion of the ſector CAM, whoſe fluent is the hyperbolic logarithm of $\frac{b+z}{b-z}$, multiplied by $\frac{1}{4}ab$. For the fluxion of the logarithm $b+z$ is $\frac{\dot{z}}{b+z}$, by *Art.* 57, and that of $b-z$, $\frac{-\dot{z}}{b-z}$, by the ſame: This laſt ſubtracted from the firſt, gives $\frac{2b\dot{z}}{bb-zz}$, which multiplied by $\frac{1}{4}ab$, gives the fluxion propoſed.

E X-

EXAMPLE.

Let $b=2z$; then $\frac{b+z}{b-z}=3$, whoſe tabular logarithm is .4771213, which multiplied by 2.302585, *Art.* 58, gives 1.0986123, for the hyperbolic logarithm of that number; and this multiplied by $\frac{1}{4}ab$, gives $.274653\times ab$, for the area of the ſector CAM.

PROBLEM.

61. *To expreſs the ſame area* CAM, *by parts of the ordinate* PM *to the firſt axis* CA.

Let the ordinate PM produced, meet the aſymptote in N, and the tangent at M, the axis in V; if PM $=y$ and the reſt as before, then CA will be to CV as the fluxion $\frac{1}{2}a\dot{y}$ of the triangle CAM, is to the fluxion of the ſector CAM, by *Art.* 23, but CP:CA ::CA:CV, by the property of the curve, as alſo CP: CA::PN:AT; and the difference between the ſquares of PN, and PM, is equal to the ſquare of AT, or $bb+yy=PN^2$; therefore by equality of ratios $\sqrt{bb+yy}:b::\frac{1}{2}a\dot{y}:\frac{ab\dot{y}}{2\sqrt{bb+yy}}$ equal to the fluxion of the ſector. If $z=\sqrt{bb+yy}$, the hyperbolic logarithm of $\frac{y+z}{b}$, multiplied by $\frac{1}{2}ab$, will be the fluent.

For the fluxion of the logarithm $\frac{y+z}{b}$ is $\frac{\dot{y}+\dot{z}}{y+z}$ by *Art.* 57; and the fluxion of $zz=bb+yy$ is $z\dot{z}=y\dot{y}$: hence $\dot{y}+\dot{z}=\dot{y}+\frac{y\dot{y}}{z}$ or $y+z\times\frac{\dot{y}}{z}$, which divided by $y+z$, gives $\frac{\dot{y}}{z}$ or $\frac{\dot{y}}{\sqrt{bb+yy}}$, whoſe product multiplied by $\frac{1}{2}ab$ is equal to the fluxion propoſed.

EXAMPLE.

If $y=\frac{1}{2}b\sqrt{3}$; then $\frac{y+z}{b}=\frac{1}{2}\sqrt{3+\sqrt{7}}=2.1889$ nearly, whoſe tabular logarithm is .3402259, and this multiplied by 2.302585, *Art.* 58, gives .391699 $\times ab$, for the area of the ſector CAM.

LEMMA. *Fig.* 30.

62. *If from the center* C *of an ellipſis, and with half the greater axis* CA *as radius, a quadrant* AND *of a circle be deſcribed, and thro' the interſections of the ſame ordinate the lines* CN, CM, *are produced ſo as to meet the tangent* AT *in* T,t; *and from the point* t *a line* tQ *be drawn parallel to* CT, *and* Am *perpendicular to it, then will* AQ *be equal to* CB *half the ſecond axis, and* Am *equal to the ordinate* PM *of the ellipſis.*

By the parallels we have PM : PN (:: At : AT) :: AQ : AC, and by the property of the ellipſis, PM : PN :: CB : CD or CA. Therefore AQ : AC :: CB : AC, by equality of ratios, and AQ = CB.

The right-angled ſimilar triangles AmQ, CPN, give AQ : CN or CB : CA :: Am : PN, and by the property of the ellipſis CB : CA :: PM : PN; therefore Am : PN :: PM : PN, by equality of ratios, and Am = PM.

PROBLEM.

63. *To expreſs the area of an elliptical ſector* CAM *by parts of the tangent* At.

Let CA = a, CB = b, At = z; then the ſquare of Ct is to the ſquare of CM, as the fluxion $\frac{1}{2}a\dot{z}$ of the triangle CAt is to the fluxion of the ſector CAM, by *Art.* 25. But Ct : CM :: CT : CN or CA, and by the right-angled ſimilar triangles CAT, QAt, we have CT : CA :: Qt : QA or CB; hence Ct : CM :: Qt : CB, by equality of ratios. Therefore $bb+zz$: bb ::

$bb :: \frac{1}{2}a\dot{z} : \frac{abb\dot{z}}{2bb+2zz}$ equal to the fluxion of the ſector CAM; whoſe fluent is a circular arc deſcribed by the radius CB (b) and tangent is At (z) multiplied by $\frac{1}{2}a$, *Art.* 50.

EXAMPLE.

Let $2z=b$; that is, the radius being 100000, the tangent will be 50000, which anſwers to an angle of 26 degrees and 34 minutes, or $26\frac{1}{3}\frac{7}{0}$ degrees nearly; this multiplied by $.017453\times b$, *Art.* 54, gives $.46358\times b$, and this by $\frac{1}{2}a$ gives $.23179\times ab$ for the required area.

PROBLEM.

64. *To expreſs the ſame ſector* CAM, *by parts of the ordinate* PM (y) *to the axis* CA.

Let the tangent at M meet CA produced in V; then CA is to CV, or CP to CA, by the property of the ellipſis, as the fluxion $\frac{1}{2}a\dot{y}$ of the triangle CAM, is to the fluxion of the ſector CAM, by *Art.* 23, and by the ſimilar triangles AmQ, CPN, CP : CN or CA :: Qm : QA; or becauſe $AQ=b$, $Qm=\sqrt{bb-yy}$, we have $\sqrt{bb-yy} : b :: \frac{1}{2}a\dot{y} : \frac{ab\dot{y}}{2\sqrt{bb-yy}}$ equal to the fluxion of the ſector; whoſe fluent is a circular arc deſcribed with the radius is b and ſine y, multiplied by $\frac{1}{2}a$, by *Art.* 22.

EXAMPLE.

Let $3y=2b$; then if the radius is 10000, the ſine y will be 66666, which anſwers to an arc of 41 degrees and 48 minutes nearly, or $41\frac{4}{5}$: this being multiplied by $.01745\times b$, *Art.* 54, gives $.72941\times b$, and therefore $.3647\times ab$, will expreſs the area required.

This method of finding fluents, by means of tables of ſines and logarithms, ſaves much trouble, when they depend on the quadratures of the conic ſection; for this could no otherwiſe be done than by infinite ſeries,

ſeries, which in moſt caſes converge but ſlowly. If the arcs or logarithms are required to a greater degree of exactneſs, the rule in *Art.* 381, of my Elements of Mathematics, muſt be applied.

We ſhall give a few general problems for finding the fluents of binomial expreſſions, and ſhew when they may be found exactly or reduced to the quadrature of the conic ſections.

LEMMA.

65. Suppoſe $z^{8n}v^{s+1}$, whoſe fluxion $8n\dot{z}z^{8n-1}v^{s+1} + \overline{s+1}, \dot{v}z^{8n}v^{s}$, may be reduced into this form, $8n\dot{z}v + \overline{s+1}, \dot{v}z \times$ by $z^{8n-1}v^{s}$.

Now, if $v = c + fz^{n}$, then $\dot{v} = nf\dot{z}z^{n-1}$, and the values of v and $\dot{v}$, wrote into the laſt fluxions, gives $8nc\dot{z} + 8nf\dot{z}z^{n} + \overline{s+1}, n\dot{z}fz^{n} \times$ by $z^{8n-1}v^{s}$, or $8e + \overline{8+s+1}, fz^{n} \times n\dot{z}z^{8n-1}v^{s}$.

PROBLEM.

66. *To find the fluent of* $\dot{z}z^{rn-1} \times \overline{e+fz^{n}}^{s}$, *when* r *is greater than unity.*

Suppoſe $v^{s+1} \times$ by $az^{rn-n} + bz^{rn-2n} + cz^{rn-3n} + dz^{rn-4n} \ldots \pm xF$ to be the fluent ſought, F being the fluent of the remainder and x its coefficient, both to be determined.

The fluxion of this fluent, being diſpoſed into a proper order, by making $r+s=q$, gives $qfa + \left.\begin{matrix}\overline{r-1}, ca \\ \overline{q-1}, bf\end{matrix}\right\} z^{-n} + \left.\begin{matrix}\overline{r-2}, eb \\ \overline{q-2}, fc\end{matrix}\right\} z^{-2n} + \left.\begin{matrix}\overline{r-3}, ec \\ \overline{q-3}, fd\end{matrix}\right\} z^{-3n} + \overline{r-4}, dz^{-4n} \times$ by $n\dot{z}z^{rn-1}v^{s}$.

For, if inſtead of 8 we write $r-1$, $r-2$, $r-3$, &c. into the fluxion $8e + \overline{8+s+1}, fz^{n} \times n\dot{z}z^{8n-1}v^{s}$, we ſhall get the fluxions of the ſeveral terms of the ſuppoſed fluent.

Now,

Now, the coefficient qfa of the firſt term being made equal to unity, and thoſe of the other terms, each equal to o; the ſuppoſed fluent will come that required.

$$qfa = 1 \qquad a = \frac{1}{qf}$$

$$\overline{r-1}, ea + \overline{q-1}, fb = o \qquad -b = \frac{r-1}{q-1} \times \frac{ea}{f}$$

$$\overline{r-2}, eb + \overline{q-2}, fc = o \qquad c = \frac{r-2}{q-2} \times \frac{eb}{f}$$

$$\overline{r-3}, ec + \overline{q-3}, fd = o \qquad -d = \frac{r-3}{q-3} \times \frac{ec}{f}$$

And becauſe $x = \overline{r-4}, edn$ and $\dot{F} = \dot{z}z^{rn-4n-1}v^s$; when $dz^{rn-4n}v^s + {}^1$, is the laſt term; by the ſame reaſon $x = negQ$ and $\dot{F} = \dot{z}z^{gn-1}v^s$, when $Qz^{gn-1}v^s + {}^1$ is the laſt term.

The coefficients above being wrote into the fluent, gives $\frac{v^{s+1}z^{rn-n}}{nqf} - \frac{r-1}{q-1} \times \frac{Ae}{fz^n} + \frac{r-2}{q-2} \times \frac{Be}{fz^n} - \frac{r-3}{q-3} \times \frac{Ce}{fz^n} + \frac{r-4}{q-4} \times \frac{De}{fz^n} \cdots \pm negQF.$

The letters A, B, C, D, &c. expreſs each its preceding term, and the remainder F muſt always have a contrary ſign to that of the laſt term of the ſeries; the coefficient or letter Q is always equal to that of the laſt term of the fluent.

C O R.

67. Hence, if r be any whole poſitive number, and q a fraction, or a whole poſitive number greater than r, or elſe when s is a whole negative number greater than unity, the fluent will be finite and contain r terms; but when r is a fraction, the fluent depends upon that of F, and in this caſe g muſt be the leaſt fraction of r.

C O R.

C O R.

68. When r is negative, the fluent of $\dot{z}z^{-rn-1} \times \overline{e+fz^n}^{s}$ may be found by the former. For if the term z^{-rn-1}, without the radical sign, be multiplied by z^{sn}, and that $e+fz^n$ divided by z^n, we get $\dot{z}z^{sn-rn-1} \times \overline{ez^{-n}+f}^{s}$, without changing its value; and as $\frac{sn-rn}{-n} = r-s$, this value wrote into that of $q=r+s$, found above, instead of r, gives $q=r$, and writing q for r in the numerator, r for q in the denominator, as well as $-n$, f, e, z^{-n} instead of n, e, f, z^n, respectively, the last fluent becomes $\frac{z^{n-qn}v^s+1}{-ner} \times$ $\frac{q-1}{r-1} \times \frac{A}{e}fz^n - \frac{q-2}{r-2} \times \frac{B}{e}fz^n + \frac{q-3}{r-3} \times \frac{C}{e}fz^n$ - - - $\pm g\,nf$ QF.

Observe we have here $v=ez^{-n}+f$ or $v=\frac{e+fz^n}{z^n}$ and $q=r-s$.

C O R.

69. Hence, if q or its equal $r-s$ is a whole number, and r a fraction or any whole positive number greater than q, the fluent will be finite, and consist of q or $r-s$ terms; and when q is a fraction, g must be the least positive fraction of r, and the series must be continued to $r-s$ terms.

E X A M P L E.

70. Let $\frac{\dot{z}z^{rn-1}}{e+fz^n}$ be the fluxion, and r any whole positive nuinber; then will F be the fluent of $\frac{\dot{z}z^{n-1}}{e+fz^n}$, which is the hyperbolic logarithm of $\frac{e+fz^n}{e}$, divided by fn, by *Art.* 57; and since s is here -1, the general fluent

ent, in *Art.* 36, becomes in this caſe $\frac{z^{rn-n}}{nqf} - \frac{r-1}{q-1} \times \frac{Ae}{fz^n} + \frac{r-2}{q-2} \times \frac{Be}{fz^n} - \frac{r-3}{q-3} \times \frac{Ce}{fz^n} \cdots\cdots \frac{e}{f}$ QF; and the ſeries muſt be continued to $r-1$ terms.

When r is negative, the fluent of $\frac{\dot{z}z^{-rn-1}}{e+fz^n}$ depends on that of $\frac{\dot{z}z^{-1}}{e+fz^n}$, which is the hyperbolic logarithm of $\frac{ez^{-n}+f}{f}$, by *Art.* 57, divided by $-ne$; and the general fluent in *Art.* 68. becomes $\frac{z^{n-qn}}{-ner} + \frac{q-1}{r-1} \times \frac{A}{e} fz^n - \frac{q-2}{r-2} \times \frac{B}{e} fz^n - \frac{q-3}{r-3} \times \frac{C}{e} fz^n \cdots \frac{f}{e}$ F.

This fluent muſt be continued to r terms.

PROBLEM.

71. *To find the fluent of* $\dot{z}z^{rn-1} \times \overline{e+fz^n}^s$ *in a different manner.*

Suppoſe $v = e+fz^n$ and $z^{rn}v^s \times$ by $a + b\frac{z^n}{v} \times c\frac{z^{2n}}{vv} + d\frac{z^{3n}}{v^3} +$ &c. to be the fluent ſought. The fluxion of $z^{rn}v^s$, will be $rn\dot{z}z^{rn-1}v^s + sz^{rn}\dot{v}v^{s-1}$, or writing the values $nf\dot{z}z^{n-1}$ of $\dot{v}$, becomes $rn\dot{z}z^{rn-1}v^s + nfs\dot{z}z^{rn+n-1}v^{s-1}$; or which is the ſame, $r + fs\frac{z^n}{v} \times$ by $n\dot{z}z^{rn-1}v^s$. Now, writing $r+1$, $r+2$, $r+3$, &c. for r, and at the ſame time $s-1$, $s-2$, $s-3$, &c. for s, this laſt fluxion will become that of the firſt, ſecond, third, &c. term of the ſuppoſed fluent, each being multiplied by its reſpective coefficient a, b, c, d, and placed in proper order, gives

$$ar+\frac{\overline{s-o},\,af}{r+1,\,b}\Big\}\frac{z^n}{v}+\frac{\overline{s-1},\,bf}{r+2,\,c}\Big\}\frac{z^{2n}}{vv}+\frac{\overline{s-2},\,cf}{r+3,\,d}\Big\}\frac{z^{3n}}{v^3}\times \text{ by } n\dot{z}z^{rn-1}v^s.$$

Now if the coefficient ar of the firſt term be made equal to unity, that of the propoſed fluxion, and each of the reſt equal to o, we get

$$ar=1, \qquad a=\frac{1}{r}$$

$$\overline{s-o},\,af+\overline{r+1},\,b=o \qquad -b=\frac{s-o}{r+1}\times af$$

$$\overline{s-1},\,bf+\overline{r+2},\,c=o \qquad c=\frac{s-1}{r+2}\times bf$$

$$\overline{s-2},\,cf+\overline{r+3},\,d=o \qquad -d=\frac{s-2}{r+3}\times cf$$

Therefore $\frac{z^{rn}v^s}{rn}-\frac{s-o}{r+1}\times\frac{A}{v}fz^n+\frac{s-1}{r+2}\times\frac{B}{v}fz^n-\frac{s-2}{r+3}\times\frac{C}{v}fz^n+$ &c. will be the fluent required: the letters A, B, C, &c. expreſs as uſual, each the term that preceeds it.

C O R.

72. Hence, when s is any whole poſitive number, and r a broken or a whole poſitive number, the fluent will be finite and expreſſed by $s+1$ terms.

And when e and f are both poſitive, this ſeries will always converge; but when e is poſitive and f negative, the former ſeries will always converge, and therefore ſhould be uſed, when it is not finite or can be reduced to the quadrature of the conic ſection.

There is beſides a method whereby this laſt ſeries may be brought to converge faſter, as follows: Let $\frac{\dot{z}}{1+zz}$ be the fluxion propoſed, which being reduced into a ſeries by a continued diviſion, and the fluent taken,

taken, gives $z-\frac{1}{3}z^3+\frac{1}{5}z^5-\frac{1}{7}z^7+$ &c. and there will remain $\frac{\dot{z}z^{2t}}{1+zz}$, t expreſſes the number of the preceding terms, the fluent of this remainder added to the preceding terms, will make up the fluent of the fluxion propoſed $\frac{\dot{z}}{1+zz}$.

Now, the fluent of $\frac{\dot{z}z^{2t}}{1+z^2}$, by the general form above, is $\pm\frac{z^{2r}}{2rv}+\frac{1}{r+1}\times\frac{z^2}{v}A+\frac{2}{r+2}\times\frac{z^2}{v}B+\frac{3}{r+3}\times\frac{z^2}{v}C+$, &c. becauſe $n=2, f=1=e$, $2r-1=2t$, or $2r=2t+1$.

But if $z=1$, the firſt ſeries becomes $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}$ &c. and the ſum of the 10 firſt terms or of 5 of this $\frac{2}{3}+\frac{2}{3\,5}+\frac{2}{9\,9}+\frac{2}{1\,9\,2}+\frac{2}{3\,2\,3}$, gives .76045,99049; and as t is 10 in this caſe, we get $r=\frac{21}{2}$, we have alſo $v=2$; and henee the ſeries of the remainder becomes $\frac{1}{4\,2}+\frac{1}{2\,3}A+\frac{2}{2\,5}B+\frac{3}{2\,7}C+\frac{4}{2\,9}D+\frac{5}{3\,1}E+$, &c. which reduced into decimals gives

.02380,95283
103,51967
8,28158
92018
12692
2047
372
75
16
4

.02493,82585

The ſum of theſe two numbers gives .78539, 81634, for an arc of a circle of 45 degrees true in all its places: whereas the firſt ſeries would require more than 1000000000 terms.

PROBLEM.

73. *To find the fluent of* $\dot{z}z^{rn-1}v^s \times$ *by* $a+bz^n+cz^{2n}+dz^{3n}+$, *&c. whence* v *becomes* o.

If A expreſſes the fluent of $\dot{z}z^{rn-1}v^s$ and B that of $\dot{z}z^{rn+n-1}v^s$: then will $ern\mathrm{A}+r+s+1, nf\mathrm{B}=z^{rn}v^{s+1}$ by *Art.* 65; becauſe 8 there is r here; if $r+s=q$, when v or its equal $e-fz^n$ becomes equal to o, we get $\mathrm{B}=\frac{r+o}{q+1}\times\frac{e}{f}\mathrm{A}$, becauſe f is here negative. By writing $r+1.r+2, r+3$, into the fluxions of A and B, they become thoſe of the ſucceeding terms C, D; thus $\mathrm{B}=\frac{r+o}{q+1}\times\frac{e}{f}\mathrm{A}$ becomes $\mathrm{C}=\frac{r+1}{q+2}\times\frac{e}{f}\mathrm{B}$, $\mathrm{D}=\frac{r+2}{q+3}\times\frac{e}{f}\mathrm{C}$. Hence if we write the value of B into that of C, we get $\mathrm{C}=\frac{r+o.r+1}{q+1.q+2}\times\frac{ee}{ff}\mathrm{A}$; and this value of C into that of D, $\mathrm{D}=\frac{r+o.r+1.r+2}{q+1.q+2.q+3}\times\frac{e^3}{f^3}\mathrm{A}$. Conſequently, theſe values multiplied by their reſpective coefficients gives A$\times$ by $a+\frac{r+o}{q+1}\times\frac{be}{f}+\frac{r+o.}{q+1.}\frac{r+1}{r+2}\times\frac{cee}{ff}+\frac{r+o.r+1.r+2}{q+1.q+2.q+3}\times\frac{de^3}{f^3}$ for the fluent required. Obſerve, 1°. That the points between the factors $r+o.q+1$, ſignify multiplication: that is, as $\overline{r+o}\times\overline{r+1}\times\overline{r+2}$, and $\overline{q+1}\times\overline{q+2}$, &c. 2°. All but one of the coefficients a, b, c, d, may be o: thus if $a=b=c=o$; then $\mathrm{A}\times\frac{r+o.r+1.r+2}{q+1.q+2.q+3}\times\frac{de^3}{f^3}$, will be the fluent.

3°. If θ expreſſes any whole poſitive number, the fluent of $\dot{z}z^{rn-1}v^s\times dz\theta^n$, will have as many factors $r+o, r+1, r+2$, and $q+1, q+2, q+3$, of both ſorts as there are units in θ, multiplied by $\frac{de\theta.}{f\theta}$.

PRO-

PROBLEM. *Fig.* 31.

74. *Let the curve* LMN *be ſuch that any ordinate* PM$=y$, *whoſe diſtance* AP *from a given point* A *is* z, *may be expreſſed by the equation* $y=A+zB+\frac{z.z-1}{1.2}C+\frac{z.z-1\ z-2}{1.2.3}D+$, &c. *to find the coefficients* A, B, C, D, &c.

In the baſe AP, take the equal parts AB, BC, CD, DE, &c. which being each to AP as unity to z, and draw ordinates thro' theſe points of diviſions, which call, a, b, c, d, e, &c.; then by making z equal to o, 1, 2, 3, 4, &c. ſucceſſively, in the equation the ordinate y becomes equal to a, b, c, d; this gives the following equalities.

$a=A$		$A=a$
$b=A+B$		$B=b-a$
$c=A+2B+C$	hence	$C=c-2b+a$
$d=A+3B+3C+D$		$D=d-3c+3b-a$
$e=A+4B+6C+4D+E$		$E=e-4d+6c-4b+a$.

Conſequently if theſe ordinates are given, the coefficients A, B, C, are given alſo.

PROBLEM.

75. *The equation of the curve* LMN *being* $y=A+zB+\frac{z.z-1}{1.2}C+$, &c. *to find the area* ALMN.

If y and its equal be multiplied by $\dot{z}$, and the fluent of each term taken as uſual, we get $z\times$ by $A+\frac{1}{2}Bz+\frac{1}{2}C\times\frac{zz}{3}-\frac{z}{2}+\frac{1}{6}D\times\frac{z^3}{4}-\frac{3z^2}{3}+\frac{2z}{2}+\frac{1}{24}E\times\frac{z^4}{5}-\frac{6z^3}{4}+\frac{11z^2}{3}-\frac{6z}{2}+$, &c. For the area required, the continuation is plain: for the numerators are formed by continually multiplying the terms of the ſeries z, $z-1$

$z-1$, $z-2$, $z-3$, &c. the diviſors of the ſeveral powers of z, by unity more than their indices and the diviſors of the coefficients A, B, C, the products of the numbers 1,2,3,4,5,6, after the ſecond term.

EXAMPLES.

76. Hence, if $z=2$, we get $z\times\overline{A+B+\frac{1}{6}C}$, nearly for the area.

If $z=3$, then $z\times$ by $A+\frac{3}{2}B+\frac{3}{4}C+\frac{1}{8}D$.

If $z=4$, $z\times$ by $A+2B+\frac{5}{3}C+\frac{2}{3}D+\frac{7}{90}E$. Or becauſe $A=a$, $B=b-a$, $C=c-2b+a$, $D=d-3c+3b-a$, $E=e-4d+6c-4b+a$, by *Art.* 74, we get $A+B+\frac{1}{6}C=\frac{c+4b+a}{6}$. $A+\frac{3}{2}B+\frac{3}{4}C+\frac{1}{8}D=\frac{d+3c+3b+a}{8}$: and $A+2B+\frac{5}{3}C+\frac{2}{3}D+\frac{7}{90}E=\frac{7e+32d+12c+32b+7a}{90}$.

Hence, if A expreſſes the ſum of the firſt and laſt ordinate, B the ſum of the ſecond and laſt but one, C that of the two next to theſe, and D, E, &c. always the ſum of the two next, till you come to the two middle ones, or if the number of terms is odd, the laſt letters denote the middle term. The following table, where the number of ordinates is expreſſed by Roman figures, will give the areas nearly, z expreſſing always the baſe or diſtance between the firſt and laſt ordinates.

III. $\frac{A+4B}{6}z$

IV. $\frac{A+3B}{8}z$

V. $\frac{7A+32B+12C}{90}z$

VI. $\frac{19A+75B+50C}{288}z$

VII. $\frac{41A+216B+27C+272D}{840}z$

VIII.

VIII. $\frac{751A+3572B+1323C+2989D}{17280}z$

IX. $\frac{989A+5888B-928C+10496D-4540E}{28350}z$.

To apply this method to ſome examples, we ſhall ſuppoſe $\frac{\dot{z}}{1+z}$ to be the fluxion whoſe fluent is required when $z=1$. Suppoſe the baſe z to be divided into ſix equal parts, viz. $\frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}, \frac{6}{6}$; then writing o and theſe parts for z into $\frac{1}{1+z}$, we get the ſeven following ordinates $1, \frac{6}{7}, \frac{6}{8}, \frac{6}{9}, \frac{6}{10}, \frac{6}{11}, \frac{6}{12}$; hence the form $\frac{41A+216B+27C+272D}{840}$, for ſeven ordinates, gives $A=1+\frac{6}{12}=\frac{3}{2}$, $B=\frac{6}{7}+\frac{6}{11}=\frac{108}{77}$, $C=\frac{6}{8}+\frac{6}{10}=\frac{27}{20}$, $D=\frac{6}{9}=\frac{2}{3}$; theſe values multiplied by their reſpective coefficients, and reduced into decimals, give

61.50000
302.96078
36.45000
181.33333
582.24411

and the ſum divided by 840 gives .693147 for the area required, which is the hyperbolic logarithm of 2, true in all its places.

Suppoſe the fluxion $\frac{\dot{z}}{1+zz}$, whoſe fluent is required, when $z=1$; divide the baſe z into five equal parts, as $\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1$; by writing o and theſe values into $\frac{1}{1+zz}$, we get theſe ſix ordinates $1, \frac{25}{26}, \frac{25}{29}, \frac{25}{34}, \frac{25}{41}, \frac{1}{2}$; now by

by the area of ſix ordinates $\frac{19A+75B+50C}{288}$, we get $A=1+\frac{1}{2}=\frac{3}{2}$, $B=\frac{25}{26}+\frac{25}{41}=\frac{1675}{1066}$, $C=\frac{25}{29}+\frac{25}{34}=\frac{1575}{986}$, and theſe values multiplied by their reſpective coefficients give

28.5000
117.8470
79.8600
226.2070

This ſum divided by the diviſor 288, gives .7854 for an arc of a circle of 45 degrees, whoſe radius is unity. Obſerve that when the baſe z is more or leſs than unity, the area found as above muſt be multiplied by it, as it is marked in the tables: obſerve alſo, that the more ordinates made uſe of, the nearer the area will be; and this method can be applied when all others fail, that is, when the ordinates are expreſſed by *ſurd* quantities, as will be ſeen hereafter.

A GENERAL METHOD of finding the ROOTS of Equations by Approximation.

MANY attempts have been made by authors to find the roots of equations; but as their operations are in general very intricate and tedious, and beſides no general rule has been given by any ſo as to anſwer all caſes, excepting *Daniel Bernoullie*, in the Memoirs at *Peterſburgh*; but as he has given no direct demonſtration, and, I think, is miſtaken in ſome caſes, the reader will be pleaſed to find here a very conciſe demonſtration, as alſo a general rule, expreſſed in fewer words, than could hardly have been expected in ſo extenſive a ſubject.

PRO-

PROBLEM.

77. *To find the leaſt value of* x *in the general equation* $1=ax+bx^2+cx^3+dx^4+$, &c.

If we divide both ſides by x, we get $\frac{1}{x}=a+bx+cxx+dx^3+$, &c. and if we ſuppoſe x ſo ſmall that all the terms multiplied by it may be neglected as inconſiderable, we get $x=\frac{1}{a}=\frac{1}{A}$ nearly. To get a nearer value of x, write $\frac{1}{A}$ for x in bx, and by neglecting the higher powers, we get $\frac{1}{x}=a+\frac{b}{A}$, or $x=\frac{A}{aA+b}=\frac{A}{B}$. To carry on the approximation, write this laſt value of x into bx and its ſquare, or which is nearly the ſame, and more convenient, the product $\frac{1}{A}\times\frac{A}{B}=\frac{1}{B}$, of the preceding values, into cx^2, and neglecting the higher powers, we get $\frac{1}{x}=a+\frac{bA+c}{B}$, or $x=\frac{B}{aB+bA+c}=\frac{B}{C}$. Writing again this laſt value of x into bx, the product $\frac{B}{C}\times\frac{A}{B}=\frac{A}{C}$, of the two laſt values of x into cxx, and the product $\frac{B}{C}\times\frac{A}{B}\times\frac{1}{A}=\frac{1}{C}$, of the three laſt values of x into dx^3, by neglecting the higher powers, if there are any more; then $\frac{1}{x}=a+\frac{bA+cA+d}{C}$, or $x=\frac{C}{aC+bB+cA+d}=\frac{C}{D}$. By continuing thus to write the laſt value found for x, the product of the two laſt for x^2, the product of the three laſt for x^3, and the product of the four laſt for x^4, and repeating theſe operations, the value of x may be approximated to any degree of exactneſs. The continuation of this rule is evident by this table.

$1=1$
$A=a$
$B=aA+b$
$C=aB+bA+c$
$D=aC+bB+cA+d$
$E=aD+bC+cB+dA.$

GENERAL RULE.

78. Find a ſeries, whoſe firſt term being unity, the ſecond, the firſt coefficient, then to find any other term, multiply conſtantly the coefficients each by the reſpective term of the ſeries, in a contrary order; that is, the firſt a by the laſt term, the ſecond b, by the laſt but one, the third c by the laſt but two, &c. and the ſum of all theſe products will be the next term of the ſeries; by continuing thus the ſeries at pleaſure, the more terms the better, then the laſt term but one divided by the laſt, will be the root required nearly.

EXAMPLE I.

Let $1=x+2x^2+3x^3+4x^4$; then will 1:1; the next term $1\times1+1\times2=3$; the next $1\times3+2\times1+3\times1=8$; the next to this $8\times1+3\times2+3\times1+4\times1=21$; and the next to this $21\times1+8\times2+3\times3+4\times1=50$, by continuing thus we get 1.1.3.8.21.50.128.323.813.2043; and the laſt term but one 813 divided by the laſt 2043, gives $x=.39729$ nearly, or $x=.3973$. To know how near the value of x has been found, find the ſeveral powers of x, and write them into the propoſed equation, and you will find how far the approximation has been carried on. Thus here I find $1=10001$.

EXAMPLE II.

Let $1=-2x+5x^2-4x^3+x^4$, be the equation, then will 1 and -2 be the two firſt terms, the third $-2\times-2+5=9$, the fourth $-2\times9+5\times-2-4\times1=$

$1=-32$, the fifth $-2\times-32+9\times5-4\times-2+1=118$; hence the ſeries will be $1.-2.9.-32.118.-434.1595$, and $x=\frac{-434}{1595}=-.272$ nearly.

If a poſitive root is required, ſuppoſe $x=2+y$; then the equation will be changed into this $1=2y+5y^2+4y^3+y^4$, and $1:2.5.24.82.306.1123$; whence $y=.272$, and $x=2.272$, nearly.

When a term of the equation is wanting, write o in its place, and proceed as before; thus if $1=2x+x^3$; write $1=2x+o+x^3$; then 1.2, will be the two firſt terms, $2\times2+o\times1=4$ the third, $2\times4+o\times2+1=9$ the fourth, $2\times9+o\times4+2=20$, and ſo 1.2.4.9.20.44.97.214.472; hence $x=\frac{214}{472}=.4534$ nearly; ſince when this value is wrote into the equation it gives $1=10000$.

If $1=o+3x^2+2x^3+x^4$, then becauſe the root found is alternately greater and leſs, by ſuppoſing $x=.5-y$, the equation will be changed into this $1=80y-120y^2+48y^3+16y^4$; and $1.80.6280.492848$. Hence $y=.012742$ and $x=.487257$, true in all its places.

When the equation is not equal to unity, obſerve the following

GENERAL RULE.

Find the neareſt whole number to the root, which may be done by inſpection or trials; ſuppoſe that number more or leſs a variable quantity, with an indetermined coefficient equal to the variable quantity of the equation, by which the equation is changed into another, then if this indetermined coefficient be made equal to the known term, the equation will by diviſion be reduced to unity, and always converge.

EXAMPLE III.

Let $2=x+x^2+x^3$, the equation, as x is leſs than unity, ſuppoſe $x=1-y$; then will $1=6y-4y^2+y^3$; and 1.6.32.169.892.4708: now 892 divided by 4708,

gives $y=.18946$, and $x=.810535$, true in all its places.

EXAMPLE IV.

Let $2=-x-x^2+x^3$; if $3-ay=x$, the equation becomes $13=20ay-8aay^2+a^3y^3$. Hence if $a=13$, according to the rule we get $1=20y-104y^2+169y^3$, by diviſion; and 1.20.296 4009. 52776, 688608. 8960977.116523452: therefore $y=.0769$, and $3-13y=x=2.0003$ nearly; the true root being 2.

Let $14=2x+x^3$, whoſe root is 2 nearly; by making $2+ay=x$, this equation will be changed to this, $2=14ay+6aay^n+a^3y^3$. Hence $a=2$, according to the rule, and $1=14y+12y^2+4y^4$. Hence, 1.14.208. 3084.45728, and $y=.067442$; therefore $x=2.134884$, true in all its places.

Laſtly, let $90=x+xx+x^3$, whoſe root is 4 nearly; let $x=4+ay$, this will change the equation to this, $6=57ay+13aay^2+a^3y^3$; and if $a=6$, it becomes $1=57y+78y^2+36y^3$, by diviſion: and 1.57.3327. 194121.11326455; ſo that $y=.017138725$, and $x=4.10283235$.

This method is alſo excellent, in numbers, when the root is required to a great many places. Thus to find the ſquare root of 26, which is 5 nearly. Suppoſe it to be $5+y$: then $26=25+10y+y^2$, or $1=10y+y^2$. Hence 1.10.101.1020.10301.104030, and $y=\frac{10301}{104030}$; therefore $5+y=5.0990195136$0, true in all its places. The ſquare root of 487 is 22 and 3 over: if $22+3y$, be the root ſought, then $487=484+132y+9y^2$, or $1=44y+3y^2$, by reduction. Hence 1.44,1939,85448,3765529, and $22+3y=22.06807, 64908$, its root required.

If the root of a ſmall number be required to a great many places, multiply that number by ſuch a ſquare number ſo as the product may be a ſquare within unity, then the root of the product divided by the root of the multiplier, gives the root required. Thus to find the

the ſquare root of 3, multiply it by 16 the ſquare of 4, which gives 48; then ſuppoſe $7-y$ to be the root: hence $48=49-14y+y^2$, or $1=14y-y^2$, by five ſteps only we get 1.73205,08075,689, for the root required.

To find the ſquare root of 2, multiply it by 144 the ſquare of 12, and ſuppoſing $17-y$ to be the root; then will $1=34y-y^2$, from whence by 4 terms the root will be 1.41921,35627.

To find the cube root of 20, which is 3 leſs 7; let $3-7y$ be the root, then will $1=27y-63y^2+49y^3$, and 1.27.666.16330.400275.9811269; hence $7y=.2855823$, and $3-7y=2.7144177$, the root required.

Laſtly, the cube root of 2 is found by multiplying it by 64 the cube of 4, and ſuppoſing it to be $5+3y$, which gives $1=75y+45y^2+9y^3$; hence 1.75.5670.428634, and $3y=.0396842$, and $5+3y$ divided by 4, gives 1.25992105. The rule holds good in any other power.

The greateſt root of an equation may be found in the ſame manner; for ſuppoſe $x^n=ax^{n-1}+bx^{n-2}+cx^{n-3}\text{---}+d$; to be the equation, and if we ſuppoſe $x=\frac{1}{y}$, we get $x^n=\frac{1}{y^n}$, $x^{n-1}=\frac{y}{y^n}$, $x^{n-2}=\frac{y^2}{y^n}$, $x^{n-3}=\frac{y_3}{y^n}$: Hence $1=ay+by^2+cy^3\text{---}dy^n$, which gives this

GENERAL RULE.

79. Let the greateſt power of the unknown quantity be on one ſide with unity for its coefficient, and all the other terms in their order on the other; find a ſeries of numbers as before; then the laſt term divided by its antecedent gives the root nearly.

EXAMPLE.

Let $x^4=5x^3+4x^2+2x+2$; then 1.5.29.167.963.55551.31999, and $x=\frac{31999}{5555}=5.7645$ nearly.

Again,

Again, let $4x^4=3x^2+x+1$; make $x=\frac{1}{y}$; then $4=3y+y^2+y^3$; or if $y=1-z$, we get $1=8z-4z^2+z^3$, and 1.8.60.449.3360.25144: hence $z=\frac{3360}{25144}$ = and $1-z=y=\frac{21784}{25144}$. Therefore $x=\frac{25144}{21784}$ $=1.15424$, nearly: for writing this value into the equation it gives $1=00001$.

Of EXPONENTIAL QUANTITIES.

When a quantity has a variable index, it is called *Exponential*, as a^z, y^x, &c.

80. *To find the fluxion of* a^x, y^x, *or* $\overline{a+y}^x$.

I. Let $a^x=z$, taking the logarithms on both ſides, which denote by the quantities with an l before them; then $\dot{x}la=lz$, whoſe fluxion is $\dot{x}la=\frac{\dot{z}}{z}$, by *Art.* 57 and multiplying both ſides, the firſt by a^x and the ſecond by z its equal, we get $\dot{x}la\times a^x=\dot{z}$ for the fluxion required.

II. Let $y^x=z$, then by logarithms $xly=lz$, whoſe fluxion is $\dot{x}ly+\frac{x\dot{y}}{y}=\frac{\dot{z}}{z}$, or becauſe $y^x=z$; this fluxion becomes $\dot{x}ly\times y^x+xy\dot{y}x^{-1}=\dot{z}$, the fluxion required.

III. Laſtly, let $\overline{a+y}^x=z$, then $xl\overline{a+y}=lz$, whoſe fluxion by logarithm is $\dot{x}l\overline{a+y}+\frac{x\dot{y}}{a+y}=\frac{\dot{z}}{z}$; or $\dot{x}l\overline{a+y}+\frac{x\dot{y}}{a+y}\times$ by $\overline{a+y}^x=\dot{z}$. From hence this rule,

Take the logarithms of the exponentials and multiply the fluxion of this logarithm by the exponential quantity.

As theſe kind of quantities occur but ſeldom in computations, we ſhall dwell no longer upon this ſubject; and only obſerve, that if the letter a in a^x denotes the number 2.71828, whoſe hyperbolic logarithm is unity; then becauſe $la=1$, the fluxion $\dot{x}la\times a^x$, of a^x, becomes $\dot{x}a^x$; on the contrary, the fluent of

$\dot{x}a^x$

$\dot{x}a^x$ will in this caſe be a^x. This remark will be uſeful hereafter.

SECTION V.

Of MOTION in General conſidered in a non-reſiſting Medium.

MECHANICS, of which the laws of motion are the foundation, in regard to the neceſſities in life, is without doubt the moſt uſeful branch of all mathematics, and at the ſame time the moſt extenſive. It is become more valuable ſince it has been applied to aſtronomy: for which reaſon the greateſt mathematicians have made it their chief ſtudy preferably to all other branches: each making uſe of what was diſcovered before, endeavoured to improve it, and to extend its limits, in ſuch a manner that this ineſtimable ſcience is arrived to ſuch a high degree of perfection, that it ſeems to be impoſſible to add any thing new worth the public notice.

Galileus diſcovered the laws of motion of bodies which fall in a non-reſiſting medium, and applied it to the curve deſcribed by a body projected with a given velocity and in a given direction.

M. *Deſcartes*, obſerving the force by which a ſtone, turned round in a ſling, endeavours to recede from the hand, opened a new field in regard to the laws of motion, much more extenſive than *Galileus* had done. Mr. *Huggens*, who called this force *centrifugal*, applied himſelf ſtrongly to the diſcovery of its laws: he after many trials, gave at laſt thirteen theorems, but concealed the demonſtrations.

As ſoon as Sir *Iſaac Newton* had knowledge of it, his extraordinary penetration led him to the diſcovery of the laws of motion, by which a moving body is drawn, or compelled out of its direction ſo as to deſcribe any curve whatever; he named this force *centripetal*: but not ſatisfied of having demonſtrated theſe laws

laws in a non-reſiſting medium, he applied them likewiſe in any medium whatever: another diſcovery much more extenſive than all that was done before him upon this ſubject.

Kepler having found, by a great many aſtronomical obſervations, that the orbits of the planets were ellipſes, and that the ſun was placed in one of the foci as center, that their periodic times were to each other as the areas deſcribed by the lines drawn from the center of the body to the center of the forces or focus; and that the ſquares of the periodic times were to each other as the cubes of their mean diſtances: the incomparable Sir *Iſaac Newton* gave mathematical demonſtrations of theſe laws; and he found beſides, that the centripetal forces which retain the body in an ellipſis were in the inverſe ratio of the ſquares of their diſtances from the focus, and that the ſame law which retains one body in its orbit, retains likewiſe all others in theirs. From thence followed the univerſal force of gravity: it was upon this principle that he compoſed his ſyſtem of the world, which certainly will render his name immortal.

Since his principles on natural philoſophy have been publiſhed, the greateſt mathematicians of Europe were occupied to unfold ſome parts of it; for he treats every thing in a very ſhort and ſuccinct manner, and gives ſinthetical demonſtrations, which renders moſt of them very obſcure: on the other hand, that book contains almoſt every part of the mathematics and natural philoſophy; ſo that there remains very little hopes of any more diſcoveries: therefore all that could be done ſince, was either to demonſtrate ſome parts in an eaſier manner, or to extend them farther than the author has done.

The intent of this ſection is, to reduce all what is diſperſed in different parts of Sir *Iſaac Newton*'s principle concerning motion, under one view, by a general ſyſtem of motion, in as clear and eaſy a manner

ner as the ſubject will admit of. All what is given depends on two general theorems only, well known by thoſe who have written upon motion; the reſt follows as neceſſary conſequences.

After having given the firſt principles of motion in a non-reſiſting medium, we endeavour to unfold the ratios of the forces which impel or attract bodies, that is, the centripetal forces, which tend towards a point, or whoſe directions are parallel; and to render this problem more conſpicuous, examples are given in the conic ſections of both caſes.

The centripetal forces, which tend towards one of the foci, being always as the ſquares of the diſtances inverſely, the times as the areas deſcribed by the line drawn from the focus to the center of the body, and the ſquares of the periodic times, as the cubes of the mean diſtances, and if theſe laws agree with *Kepler*'s obſervations, in regard to the planets, it was reaſonable to conclude that the celeſtial bodies deſcribe ellipſes in their revolutions about the ſun; which occupies one of the foci. Nevertheleſs this uniformity of the laws obſerved by *Kepler*, and thoſe deduced from the theory, ſome mathematicians have doubted, whether there might not be ſome other curve that had the ſame property as the ellipſis. Tho' it appeared to be impoſſible, yet we gave the inverſe of this problem; that is, the centripetal force being as the ſquares of the diſtances inverſely, to find the curve deſcribed by the body, which is found again to be a conic ſection, that is an ellipſis, for it cannot be a parabola or hyperbola, becauſe they open or recede from the axis continually.

It has been found by the late obſervations on the tranſit of Venus, that the horizontal parallax of the ſun was 9.732 ſeconds, from whence the mean diſtance from the earth to the ſun is found, and expreſſed by ſemidiameters of the earth, and thence the diſtances of the other planets are found by proportion.

Now, knowing the apparant diameters of the planets by obſervation, as well as their periodic times, it is eaſy to find their true diameters and ſolid contents, as well as the denſities of thoſe which have ſatellites; as for thoſe that have none, we have no method to find theirs.

Tho' our method is general, and may be applied to all the different laws of which the centripetal force may be endowed with, yet we have given no other examples than when they are as the ſquares of the diſtances inverſely, as being the only one true in nature, for the ſake of him who endeavours to inſtruct himſelf: for which reaſon we have here given ſuch queſtions only as are either very uſeful or inſtructive; and we think to have not a little obliged the learner, to ſave his time and the trouble of reading a large volume.

THEOREM. *Fig* 32.

80. *The fluxion of the velocity generated by a power varying according to any law, is as the rectangle made by the fluxion of the time elapſed, and the meaſure of that power at that time.*

Let AM be ſuch a curve, that when the abſciſſa AP denotes the time, the correſponding ordinate PM ſhall denote the meaſure of the power: then becauſe the effects are proportional to their adequate cauſes, the inſtantaneous velocities are as the meaſures of the powers that generates them; and ſince the area AMP contains all the meaſures of the power during the time AP, that area will alſo expreſs the ſum of all the velocities generated in that time. Conſequently, the fluxion of that area is equal to the fluxion of the velocity: therefore the fluxion of the velocity is equal to the rectangle made by the fluxion of the time AP, and the meaſure PM of the power at that time, by *Art.* 16.

THEO-

THEOREM.

81. *The fluxion of the ſpace deſcribed by a motion varying according to any law, is as the rectangle made by the fluxion of the time elapſed, and the velocity at that time.*

If the ordinate PM denotes the velocity at the end of the time AP; then by the ſame argument as in the laſt theorem, the area AMP will denote the ſum of all the velocities, as well as the total ſpace deſcribed by theſe velocities, during the time AP; and therefore the fluxion of that area, or the rectangle made by the fluxion of the time AP and the velocity PM, will expreſs the fluxion of the ſpace deſcribed during that time.

COR.

82. Hence, if p expreſſes the meaſure of the variable power, v the velocity generated, in the time t and z the ſpace deſcribed; then will $\dot{v}=p\dot{t}$ by *Art.* 80, and $\dot{z}=v\dot{t}$, by *Art.* 81. If we expell $\dot{t}$, by means of theſe two equations, we get $p\dot{z}=v\dot{v}$, and if $\dot{t}$ be ſuppoſed conſtant whilſt $\dot{z}$ and v vary; the fluxion of $\dot{z}=v\dot{t}$ will be $\ddot{z}=\dot{v}\dot{t}$, or $\ddot{z}=p\dot{t}\dot{t}$, by writing $p\dot{t}$ inſtead of its equal $\dot{v}$.

EXAMPLE.

83. Let the motion be uniform; then will the velocity v be uniform, and the fluent of $z=v\dot{t}$ will be $z=tv$. From whence it follows, 1°. That the ſpaces deſcribed uniformly at the ſame time are as the velocities. 2°. The ſpaces deſcribed uniformly with the ſame velocity, are as the times. 3°. Laſtly, ſpaces deſcribed uniformly with unequal velocities, and in different times, are as the velocities and times conjointly.

EXAMPLE.

84. If the generating power be conſtant, ſuch as gravity near the ſurface of the earth the fluent of $\dot{v}=p\dot{t}$, will be $v=pt$, that of $p\dot{z}=v\dot{v}$, $2pz=vv$: and

expelling v we get $2z=ptt$; in the ſame manner, by expelling p, we get $2z=vt$. Hence,

1°. The velocities acquired by a falling body freely from a ſtate of reſt, and being acted upon by gravity only, are by $v=pt$ as the times elapſed.

2°. The ſpaces deſcribed in the deſcent, are by $2zp=vv$, as the ſquares of the velocities, or by $2z=ptt$, as the ſquares of the times; or elſe by $2z=vt$, as the times and velocity conjointly.

3°. Laſtly, The ſpace deſcribed uniformly, with the velocity acquired by a body falling freely from a ſtate of reſt, is double the ſpace deſcended thro' by the body. For ſince $z=tv$, by *Art.* 83, when the motion is uniform, and $2z=tv$, when accelerated, and as v and t are the ſame; the ſpace z deſcribed uniformly is double the ſpace z deſcended thro' by the accelerated motion in the ſame time.

REMARK.

As it is not readily conceived, how the meaſures of powers, velocities and time, can be compared together, it is neceſſary to obſerve that the time denotes the order in which the other quantities are generated or produced; thus one hour, minute or ſecond, being taken for unity, then that unite is to any number of unites as one number to any other number, and the ſpaces deſcribed by falling bodies, as well as the velocities acquired, are in certain ratio's expreſſed by numbers, and therefore are as one line to another line; and thus we are enabled to compare theſe ſeveral quantities together, in as clear and diſtinct a manner as if they were of the ſame kind, as will appear by what follows.

COR.

85. If in the equation $2z=ptt$, *Art.* 84, we ſuppoſe the time t to be one ſecond, then will $2z=p$. Which ſhews that the force of gravity p is meaſured by twice the ſpace fallen thro' freely from a ſtate of reſt, in a ſecond: hence when the ſpace fallen thro' is given, the

the time during the fall is alſo given, and the contrary. Thus the ſpace fallen thro' in a ſecond is found by experiments, to be 16 feet nearly; then $p=32$ feet; and if the ſpace z be 100 feet, then $2z=ptt$, gives $100=16tt$, whoſe ſquare root is $10=4t$, or $2.5=t$. Again, ſuppoſe t to be 6 ſeconds, then $z=16\times 36$, or $z=576$ feet.

C O R.

86. Again, the equation $2pz=vv$; gives $\sqrt{2pz}=v$; or if h be the height fallen thro' in a ſecond, then $p=2h$; and $2\sqrt{hz}=v$: Which ſhews, that twice the ſquare root of the product of the ſpace fallen thro' in a ſecond, and the ſpace fallen thro' in any time, will expreſs the number of feet deſcribed uniformly by the velocity acquired in the fall, in a ſecond. For ſince the ſquares of the velocities are as the ſpaces fallen thro', *Art.* 84 N°. 2. $\sqrt{h}$ will be to $\sqrt{z}$, as the velocity acquired by falling thro' the height h to the velocity acquired thro' the height z: and the velocity $\sqrt{h}$ is to the velocity $\sqrt{z}$, as $2h$ twice the ſpace deſcribed uniformly in a ſecond is to the ſpace $2\sqrt{hz}$ deſcribed uniformly at the ſame time, *Art.* 48. N°. 3.

E X A M P L E.

87. To find the number of feet deſcribed uniformly per ſecond, by a velocity acquired in falling thro' a ſpace of a 100 feet: then $z=100$, and $h=16$, we get $hz=1600$ feet, whoſe ſquare root 40 taken twice gives 80 for the ſpace deſcribed.

Of CENTRAL FORCES.

WHEN a body is continually attracted towards a point fixed, by a force acting according to any given law, that force is called *centripetal*; but when a body continually recedes from a given point, that force is called *centrifugal*.

Many

Many objections have been made at firſt againſt the word *attraction*, as if it implied an occult quality; whereas it means no more than the force by which the bodies are retained in their orbits. Whether it proceeds from the preſſion of ſome other matter, or from an inherent quality of matter, which cauſes revolving bodies to tend towards a point fixed, is not material, for it is certain, that there is ſuch a power in nature, otherwiſe bodies would move in a right line, tangent to the curve deſcribed, by the projectile force; and therefore it is indifferent by what name it is called, provided its meaning is underſtood. The French and German mathematicians, in order to avoid this word, have ſubſtituted that of *ſolicited*, which is far from conveying any true idea of its meaning, and ſeems to be coined merely for the ſake of contradiction; and, what muſt aſtoniſh an Engliſh reader, ſeveral of our beſt authors have made uſe of this expreſſion.

GENERAL PROBLEM. *Fig.* 33.

88. *If a body projected from a given point* A *in a given direction* AE, *with a given velocity, be attracted by a force which acts according to any law, towards a point fixed* C; *to find the nature of the curve deſcribed by the body.*

From any point M in the curve let a tangent be drawn, and from the point fixed C, the perpendicular CS to it. From any point T in the tangent, draw TR perpendicular to the ray CM: then if P denotes the centripetal force at M, c the given velocity at A, v the variable velocity at M, t the time of deſcribing the arc AM; and we call AM$=z$, CM$=y$, CS$=s$; then MT$=\dot{z}$, MR$=\dot{y}$ TR$=\dot{x}$, *Art.* 4; and $\dot{z}=vt$, *Art.* 82. Now by the ſimilarity of triangles we have CM to MS or TM($\dot{z}$) to MR($\dot{y}$) as the centripetal force P in the direction CM to its effect $\frac{\dot{y}}{\dot{z}}$P, in the direction TM of the tangent; this force $\frac{\dot{y}}{\dot{z}}$P being

wrote

wrote for p into $\dot{v}=p\dot{t}$, *Art.* 82, gives $P\dot{y}\dot{t}=\dot{v}\dot{z}$, or expelling $\dot{t}$, by means of the equation $\dot{z}=v\dot{t}$, *Art.* 82. we get $P\dot{y}=v\dot{v}$.

Again, CM is to CS, as the force P in the direction CM is to its effect $\frac{s}{y}P$ in the direction CS perpendicular to the tangent; and by the ſimilarity of triangles TM:MR::CM:MS$=\frac{y\dot{y}}{\dot{z}}$; and the radius MS is to the radius MT as the fluxion $\dot{s}$ of the circular arc deſcribed by the point S is to the fluxion $\frac{\dot{s}\dot{z}^2}{y\dot{y}}$, of the circular arc deſcribed by the point T: but this laſt ſpace is deſcribed by the force $\frac{s}{y}$ P, in the direction CS: if therefore we write $\frac{\dot{s}\dot{z}^2}{y\dot{y}}$ for $\ddot{z}$ and $\frac{s}{y}$ P for p, into the equation $\ddot{z}=p\dot{t}\dot{t}$, *Art.* 82, we get $\frac{\dot{s}\dot{z}^2}{y\dot{y}}=\dot{t}\dot{t}\times\frac{s}{y}$ P, or becauſe $\dot{z}=v\dot{t}$, by expelling $\dot{t}\dot{t}$, we ſhall have $P\dot{y}s=v\dot{v}\dot{s}$.

By means of theſe two equations, we ſhall determine the nature of the curve when the law of the centripetal force P is given.

N.B. We uſed the equation $\ddot{z}=p\dot{t}\dot{t}$, to find $P\dot{y}s=v\dot{v}\dot{s}$, becauſe the ſpace $\frac{\dot{s}\dot{z}^2}{y\dot{y}}$ deſcribed by the force in the direction CS perpendicular to the tangent, is a ſecond fluxion in reſpect to the ſpace $\dot{z}$ deſcribed in the direction of the tangent; beſides, if any other expreſſion had been uſed, the equality of the fluxions would not have been preſerved. Obſerve likewiſe, that when y increaſes while the velocity v decreaſes, the firſt equation muſt be $-P\dot{y}=v\dot{v}$.

COR.

C O R.

89. Since we have always $\pm P\dot{y}=v\dot{v}$, whatever line or curve the body deſcribes, it is manifeſt, that the velocities generated by the ſame centripetal force P, are always equal at the ſame diſtance y from the point fixed C; and theſe generated by different centripetal forces, are as theſe forces at equal diſtances.

C O R.

90. Hence if P be expelled by means of the equations $-P\dot{y}=v\dot{v}$ and $P\dot{y}s=vv\dot{s}$, we get $-v\dot{s}=s\dot{v}$, or $v\dot{s}+\dot{v}s=o$, whoſe fluent is $vs=A$. If $CE=d$, be perpendicular to the tangent at A; then when v becomes c, s becomes d; and hence $cd=A$, or $vs=cd$. Which ſhews, *that in any concave curve towards the point fixed* C, *the velocity* v *is every where inverſely as the perpendicular drawn to the tangent, let the centripetal forces be what they will.*

C O R.

91. If v be expelled by means of the equations $\dot{z}=v\dot{t}$ and $vs=cd$, we get $\dot{z}s=cd\dot{t}$; and as CM:CS:: TM:TR, or $\dot{z}s=y\dot{x}$; hence $y\dot{x}=cd\dot{t}$; and ſince $y\dot{x}$ is the fluxion of the area 2CMA, we have $2CMA=cdt$. Conſequently, *in any concave curve towards the point fixed* C, *the times are always as the areas deſcribed by the ray* CM *drawn to the point of contact, let the centripetal force be what it will.*

C O R.

92. If the center C be removed at an infinite diſtance from the vertex A, ſo that the rays CA, CM, may become parallel and equal; then if the ſign of the angle CAE be k and the radius unity, then CA (y):CE(d)::1:k, or $d=ky$; hence $s\dot{z}=y\dot{x}$ found above becomes $ks\dot{z}=d\dot{x}$; whoſe fluxion is $k\dot{s}\dot{z}=d\ddot{x}$, by ſuppoſing $\dot{z}$ conſtant: now the values of s and $\dot{s}$ wrote into the equations $P\dot{y}s=vv\dot{s}$, $vs=cd$; give $P\dot{y}\dot{x}=vv\dot{x}$ and $vx=ck\dot{z}$; and as $y\dot{x}=c(d)\dot{t}$, by the laſt, and $d=ky$, we get $\dot{x}=ck\dot{t}$, or $x=ckt$.

The

The ſame otherwiſe. Fig. 34.

Let the ordinate PM be perpendicular to the baſe AB, draw RS perpendicular to the tangent MT, call PM=y, AP=x, and the reſt as before; then the force P in the direction MP is to the effect in the direction RS perpendicular to the tangent, as MR to RS, or as MT($\dot{z}$) to RT ($\dot{x}$), by the ſimilarity of triangles; which therefore is $\frac{\dot{x}}{\dot{z}}$P: and if r expreſſes the radius of curvature CM: then the radius CM is to the radius MT, as the fluxion $\dot{z}$ of the arc deſcribed by the firſt radius is to $\frac{\dot{z}^2}{r}$ the fluxion of the arc deſcribed by the ſecond, or the fluxion of the ſpace deſcribed by the power $\frac{\dot{x}}{\dot{z}}$P. Whence $\frac{\dot{x}}{\dot{z}}$P$\times ii=\frac{\dot{z}^2}{r}$, by *Art.* 82, or becauſe $\dot{z}=v\dot{t}$, we get P$r\dot{x}=vv\dot{z}$. Now ſince $r=\frac{\dot{y}\dot{z}}{\ddot{x}}$, by *Art.* 42, N°. 2, when $\dot{z}$ is conſtant the laſt equation becomes P$\dot{y}\dot{x}=vv\ddot{x}$, the ſame as above: we have likewiſe $-r=\frac{\dot{y}\dot{z}^2}{\dot{x}\,\ddot{z}}$, by the ſame *Art.* 42, N°. 2, when $\dot{x}$ is conſtant and $\dot{z}\,\ddot{z}=\dot{y}\ddot{y}$, the equation P$r\dot{x}=vv\dot{z}$; becomes P$\dot{z}^2+vv\ddot{y}=o$, in this caſe.

COR. *Fig.* 35.

94. Hence if P be expelled by the equations $-$P$\dot{y}=vv\dot{v}$ and P$\dot{y}\dot{x}=vv\,\ddot{x}$, we ſhall have $v\ddot{x}+\dot{v}\dot{x}=o$, whoſe fluent is $v\dot{x}=A\dot{z}$, becauſe $\dot{z}$ was ſuppoſed conſtant: but the coſine k of the angle EAP, is to the radius 1, as $\dot{x}:\dot{z}$ or $k\dot{z}=\dot{x}$ at A; and as $v=c$, in that point, we get $A=ck$; and the equation $v\dot{x}=A\dot{z}$, becomes $v\dot{x}=ck\dot{z}$, the ſame as above: from this equation and $\dot{z}=v\dot{t}$, we get $\dot{x}=ck\dot{t}$ or $x=ckt$.

COR.

95. Hence it appears from the equation $\frac{v\dot{x}}{\dot{z}}=ck$; that the velocity v reduced into the horizontal direction AK, is conſtant and expreſſed by ck, and in the direction of the tangent AE, by conſtant velocity c.

EXAMPLE. *Fig.* 36.

96. Let a body deſcend upon an inclined plane AB, freely from a ſtate of reſt by the force of gravity: then the velocity acquired in the deſcent is equal to the velocity acquired by falling thro' the height AC of that plane; for the tangent is here at right angles, and by *Art.* 95. the velocity equal to c that by which the body is projected upwards. And ſince ſpaces deſcribed by the ſame velocity are as the times, the time of the deſcent is to the time of the fall as AB is to AC: or if CD be perpendicular to AB; then as AB: AC::AC:AD, the time of the deſcent is to the time of the fall as AC is to AD.

COR. *Fig.* 37.

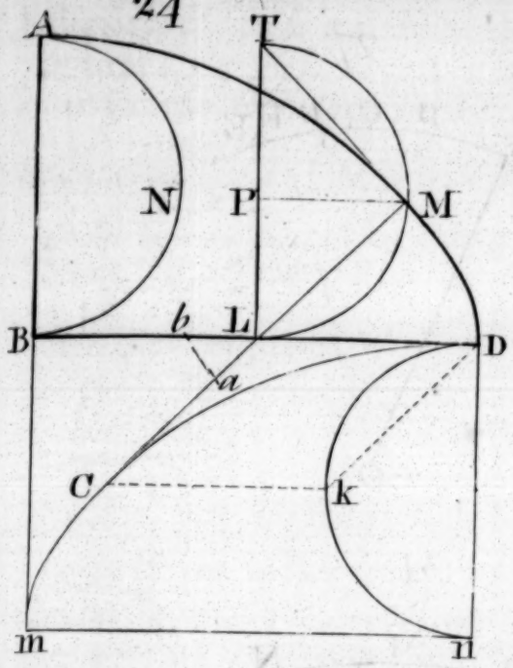
97. Hence, if from the extremity A of a vertical diameter AE, any chords AB, AC, AD, are drawn in a ſemicircle, the body will deſcend thro' either of theſe chords in the ſame time that it would fall thro' the diameter AE, by the laſt; and the velocity acquired in deſcending thro' any number of contiguous inclined planes, is always equal to the velocity acquired by falling thro' the vertical height of all theſe planes.

EXAMPLE. *Fig.* 35.

98. Let the force P be conſtant and its directions parallel; by producing the ordinate PM, ſo as to meet the tangent AE in E, and calling l the ſine of the angle PAE, the reſt the ſame as before; then whilſt the body moves uniformly in the direction AE from A to B, with the given velocity c, it will deſcend from

Plate III.

24



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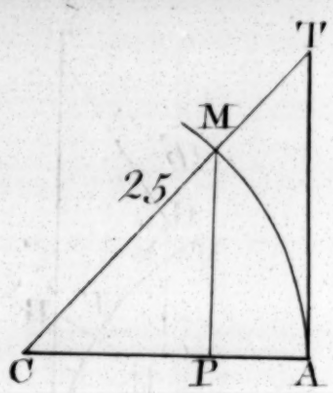

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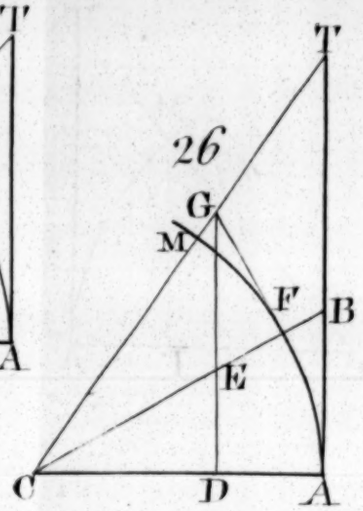

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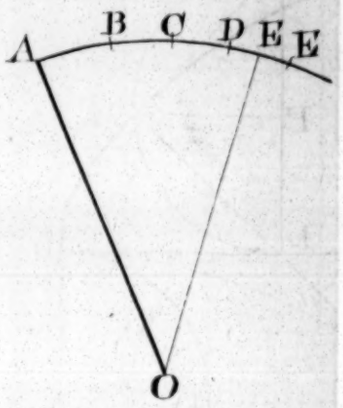

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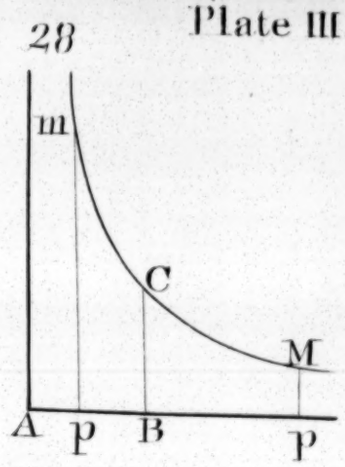

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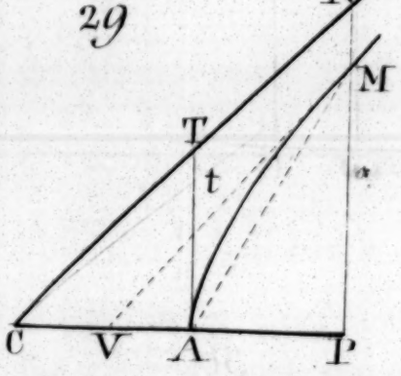

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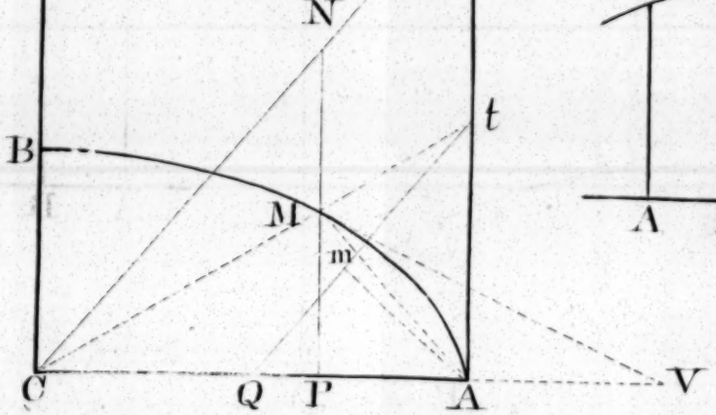

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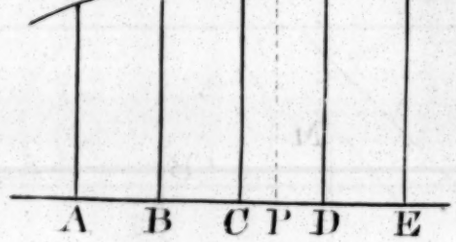

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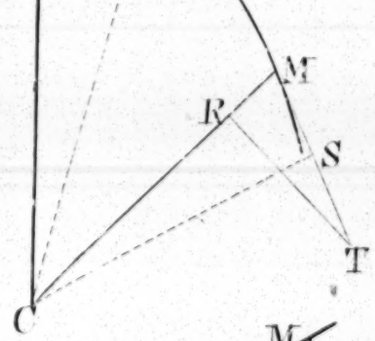

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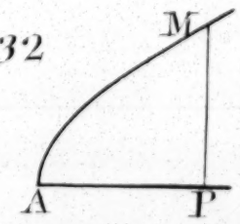

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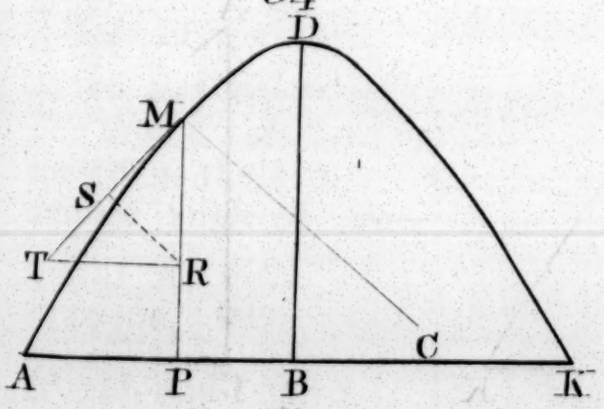

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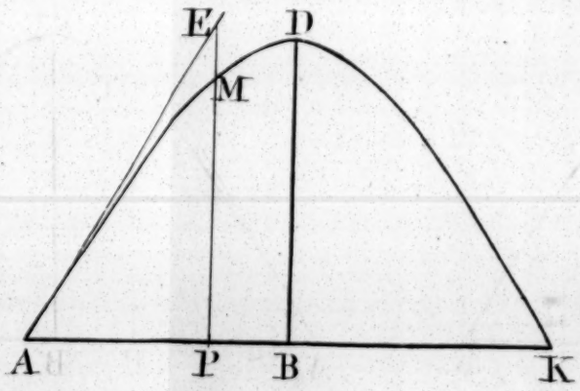

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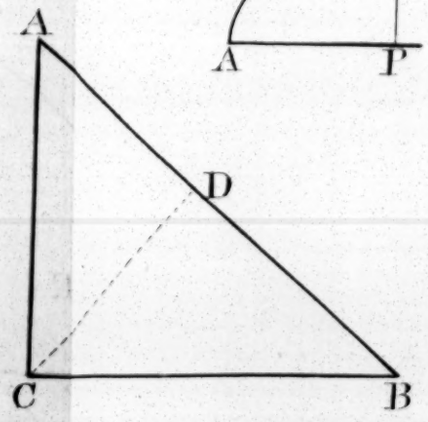
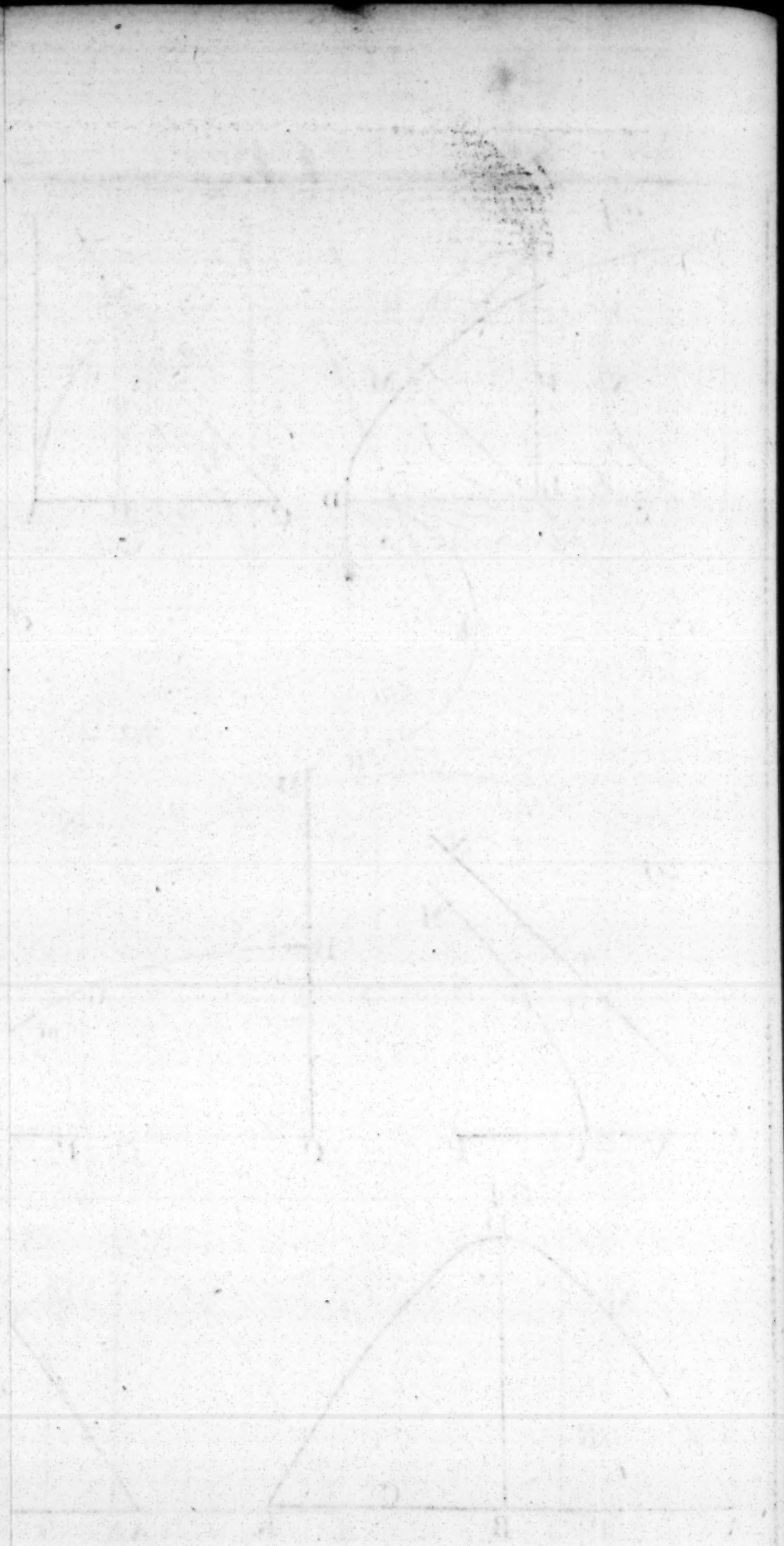




from E to M, by the force of gravity, and ſince ſpaces fallen thro' from a ſtate of reſt are as the ſquares of the times, by *Art.* 84, if d denotes the height fallen thro' in the firſt ſecond, and $x=ckt$, by *Art.* 94, we have $1:\frac{xx}{cckk}::d:\text{EM}=\frac{dxx}{cckk}$; and $k:l::x:\frac{lx}{k}=\text{PE}$.

Hence $\text{PE}-\text{EM}=\text{PM}$, or $y=\frac{lx}{k}-\frac{dxx}{cckk}$. Or if h denotes the height thro' which a falling body acquires the projectile velocity c, or $4dh=cc$, *Art.* 85; we get $y=\frac{lx}{k}-\frac{xx}{4hkk}$.

C O R.

99. If we make $y=o$, we get $\text{AK}=x=4hlk$, for the horizontal range; and if $\dot{y}=o$, $\text{AB}=x=2hlk$, for the abſciſſa correſponding to the greateſt ordinate $\text{BD}=hll$. Hence,

1°. The horizontal ranges $4hlk$, with the ſame projectile velocity $2h$, are as the ſines $2lk$ of an angle double the angle of elevation.

2°. The horizontal ranges, with the ſame angle of elevation, are as the projectile forces.

3°. The abſciſſa AB correſponding to the greateſt ordinate, is half the horizontal range AK.

C O R.

100. It is manifeſt, that the horizontal range is the greateſt, with the ſame projectile force, when the angle of elevation is 45 degrees; ſince the product lk is the greateſt when $l=k$; and all ranges equally above and below 45 degrees are equal; ſince the ſine in one caſe becomes the coſine in the other, and the coſine the ſine. Thus let $h=5280$ feet, $d=16$, and the angle of elevation 45 degrees: then $l=k$, $2lk=1$ or $k=\frac{1}{2}\sqrt{2}$. Hence $\text{AB}=5280$ feet, $\text{AK}=10560$, $\text{BD}=2640$, $c=2\sqrt{dh}=581.3$ feet per ſecond, and $t=\frac{x}{ck}=25.68$ ſeconds.

Geometrical Conſtruction. Fig. 38.

On the baſe AK, erect AL perpendicular and equal to h, deſcribe the ſemi-circumference of a circle AEeL, interſecting the direction AT in E; thro' the point E draw ED perpendicular to AK; then AK$=4$AD will be the range, and BF drawn perpendicular to the middle of AK and equal to DE, will be the greateſt height of the projectile.

For the angles ALE, DAE, which inſiſt on the ſame arc being equal, the right-angled triangles LEA, ADE are ſimilar; therefore the radius is to the ſine l as LA(h) is to AE$=hl$, and the radius is to the coſine k as AE(hl) is to AD$=hlk$; hence 4AD$=4hlk$ $=$AK. Laſtly, the radius is to the ſine l as AE(hl) is to DE$=$BF$=hll$. When the range AK and the height AL are given, the perpendicular DE determines the angle of elevation DAE.

If the object be placed above or below the level of the battery, and q the tangent of the angle it makes with the level: then as $1:q::x:y=qx$; the equation $y=\frac{lx}{k}-\frac{xx}{4hkk}$, becomes $\pm q=\frac{l}{k}-\frac{x}{4hkk}$; or if a be the tangent and s the ſecant of the angle of elevation, we get $a=\frac{h}{k}$, $s=\frac{1}{k}$, and $x=a\mp q\times\frac{2h}{ss}$

EXAMPLE. *Fig.* 39.

101. Let DMA be half a cycloid, BD its baſe placed horizontally, AB the diameter of its generating circle; it is required to find the time of deſcent thro' any arc AM.

Draw MP perpendicular to AB, and join the chord AN; then as the arc AM of the cycolid is double the chord AN, by *Art.* 44, and the velocity acquired in the deſcent thro' the arch, MA is equal to that acquired in the fall thro' the vertical height AP, *Art.* 97; but this velocity is as the ſquare root of AP,

AP, *Art.* 84, N°. 2; or becauſe AP:AN::AN:AB, the ſquare root of AP is as the chord AN, that is as the arc AM. Since the velocity is as the ſpace deſcribed, the time of deſcending thro' any arc of the cycloid, terminated by the loweſt point A, is conſequently equal, let that arc be what it will.

C O R.

102. Hence, if AP$=y$, AB$=a$, and d the height fallen thro' in a ſecond; then $-2d\dot{y}=v\dot{v}$, *Art.* 88, whoſe fluent is $A-4dy=vv$; and when $y=a$, v is $=o$, and $A=4ad$; therefore $2\sqrt{ad-dy}=v$: but $z=2\sqrt{ay}$, by *Art.* 44, or $\dot{z}=\frac{a\dot{y}}{\sqrt{ay}}$; theſe values of v and $\dot{z}$ wrote into $\dot{z}=vi$, *Art.* 82, gives $\frac{a\dot{y}}{2\sqrt{ay-yy}}=i\sqrt{ad}$. If c denotes an arc of a circle whoſe diameter is a and the ſine $\sqrt{ay-yy}$; the fluent of this fluxion will be $c=t\sqrt{ad}$, *Art.* 4, N°. 2, and when $y=o$, c expreſſes half the circumference and t the time of deſcent thro' half the cycloid DMA.

C O R.

103. Hence, if the time of the deſcent thro' half the cycloid be half a ſecond, then $c=t\sqrt{ad}$ becomes $c=\frac{1}{2}\sqrt{ad}$, or $4cc=ad$; or $2a:2tt::aa:4cc$. Conſequently, when the vibrations are performed in a ſecond, the length $2a$ of the pendulum is to the force of gravity $2d$ as the ſquare of the diameter a is to the ſquare of the circumference $2c$.

C O R.

104. Since $c=t\sqrt{ad}$ gives $cc=ttad$ or $c\sqrt{\frac{a}{d}}=at$, and $\sqrt{\frac{a}{d}}$ denotes the time of deſcent thro' the diameter AB, *Art.* 84, N°. 2; it follows, that the time of deſcending

deſcending thro' the diameter BA, is to the time of one vibration in a cycloid as the diameter a is to the circumference $2c$.

C O R.

105. Becauſe $c=t\sqrt{ad}$, gives $a\times\frac{cc}{ac}=dit$, and the ratio of the diameter to its circumference is conſtant, as well as d in the ſame latitude: the times of vibrations by pendulums of different lengths, are in a ſubduplicate ratio to that of their lengths.

C O R.

106. If a and b are the lengths of two pendulums which perform their vibrations in the time t and T; then by the laſt $\sqrt{a}:\sqrt{b}::t:\mathrm{T}$, or $\mathrm{T}\sqrt{a}=t\sqrt{b}$; and if we ſuppoſe the firſt performs n number of vibration whilſt the ſecond performs but one; then $nt=\mathrm{T}$, and $\mathrm{T}\sqrt{a}=t\sqrt{b}$, becomes $n\sqrt{a}=\sqrt{b}$, or $nna=b$. Conſequently, if one of theſe lengths is given the other is given likewiſe.

C O R.

107. Laſtly, ſince 3.14159, &c. denotes the circumerence of a circle, whoſe diameter is unity, *Art.* 53, the equation $a\times\frac{cc}{aa}=d$, (*Art.* 104.) t being unity gives $9.869a=d$, when the vibrations are performed in a ſecond. Hence, if the length of the pendulum is given, the ſpace fallen thro' in a ſecond is given likewiſe and the contrary. Dr. *Hally* has found by experiment, that the length of ſuch a pendulum is 39.125 inches in the latitude of *London*; whence, if $2a=39.125$, then $d=193.0623$ inches, or $d=16.088$ feet; for the height fallen thro' in a ſecond.

Mr. *de Mairand* has found that length to be 440.9778 lines at *Paris*; ſo that if $2a=440.567$, we get $d=2173.9778$ lines, or $d=15.097$ feet for that ſpace.

The

The length of ſuch a pendulum is commonly ſuppoſed to be 39.2 inches at *London*, which is a miſtake, as will be ſhewn hereafter.

EXAMPLE. *Fig.* 40.

108. Let the body revolve in a parabola AM, the point fixed, C the focus and A the vertex, to find the force P. If CA$=a$, CM$=y$; then CT$=s=\sqrt{ay}$, by the nature of the curve; whence the equation $vs=cd$, in *Art.* 92, becomes $ac=v\sqrt{ay}$, becauſe d is here a; or $vv=\frac{acc}{y}$, whoſe fluxion is $-v\dot{v}=\frac{acc\dot{y}}{2yy}$, and ſince $-P\dot{y}=v\dot{v}$, by *Art.* 88, we get $P=\frac{acc}{2yy}$, and at the vertex A where $y=a$, we get $P=\frac{cc}{2a}$

EXAMPLE. *Fig.* 41.

109. Let the body revolve in an ellipſis AM about one of its foci C; to find the centripetal force P. From the other focus F draw FM, and let a, b, denote half the firſt and ſecond axis; then CT$=s=\frac{by}{\sqrt{2ay-yy}}$, by the nature of the curve. Whence $vs=cd$ becomes $b\dot{v}y=dc\sqrt{2ay-yy}$, or $vv=\frac{ddcc}{bb}\times\frac{2a-y}{y}$, whoſe fluxion is $-v\dot{v}=\frac{addcc\dot{y}}{bbyy}$ equal to $-P\dot{y}$, or $P=\frac{addcc}{bbyy}$. Which ſhews, that the centripetal force of a body revolving in an ellipſis or parabola, about the focus, is as the ſquare of the diſtance inverſely.

EXAMPLE. *Fig.* 42.

110. Let the point fixed C be the center of an ellipſis; then as the ſum of the ſquares of two ſemiconjugate diameters CM and CN is conſtant, ſuppoſe $=rr$; we have CN$=\sqrt{rr-yy}$, and CN$\times$CT$=ab=s\sqrt{rr-yy}$, by the property of the curve: therefore the

the equation $vs=cd$, becomes $bv=c\sqrt{rr-yy}$, becauſe $d=a$ here. Hence $vv=\frac{cc}{bb}\times rr-yy$, whoſe fluxion is $-v\dot{v}=\frac{ccy\dot{y}}{bb}=P\dot{y}$ or $P=\frac{ccy}{bb}$; the force P is therefore as the diſtance directly.

C O R.

111. If $a=b$, then the elipſis becomes a circle and $P=\frac{cc}{a}$. Which ſhews, that the centripetal force is as the ſquare of the velocity directly and the radius inverſely; and the velocity, as the ſquare root of the rectangle made by the force and radius, likewiſe the radius, as the ſquare of the velocity directly and the force inverſely.

P R O B L E M. *Fig.* 41.

112. *If the centripetal force be as the ſquare of the diſtance inverſely, to find the nature of the curve deſcribed.*

Becauſe $-P\dot{y}=v\dot{v}$, by *Art.* 88, and $P=\frac{1}{yy}$ by ſuppoſition, we get $-\frac{\dot{y}}{yy}=v\dot{v}$, whoſe fluent is $A+\frac{2}{y}=vv$. Now if the point G be taken in the vertical line CA produced, ſuch that when CM (y) becomes CG or $2a$, the velocity v becomes o; then will $A+\frac{2}{y}=o$, or $-A=\frac{1}{a}$, and ſo $\frac{2}{y}-\frac{1}{a}=vv$; if $CA=d$, and the tangent at A be at right angles to CA, when y becomes $=d$, then v becomes $=c$, and the laſt equation $\frac{2}{d}-\frac{1}{a}=cc$. Now theſe values of vv and cc being wrote into $vvss=ccdd$, the ſquare of $vs=cd$, gives $ss\times\frac{2a-y}{y}=2ad-dd$, or if $2ad-dd=bb$, we get

$by=$

$by=s\sqrt{2ay-yy}$, which is the equation of an ellipſis in reſpect to the foci; a, b, are half the firſt and ſecond axes.

PROBLEM.

113. *To determine the periodic time of a body revolving in an ellipſis about one of its foci.*

Let T expreſs that time, a, b, half the firſt and ſecond axis, and $r=3.14159$, &c. that is, let r denote the circumference of a circle whoſe diameter is unity; then becauſe $2\text{CAM}=cdt$, by *Art.* 90, and when the area CAM becomes that of the whole ellipſis, it is $=rab$, then t becomes T, and therefore $2abr=cd\text{T}$.

COR.

114. Hence if a, b, d, become equal, the ellipſis degenerates into a circle, and $2ra=c\text{T}$; and as $a\text{P}=cc$ by *Art.* 111; the ſquare $4rraa=cc\text{TT}$ of $2ra=c\text{T}$ gives $4arr=\text{PTT}$. Which ſhews, 1°. That the centripetal forces in circles, are as the radius directly, and the ſquare of the periodic time inverſely.

2°. The radius is as the centripetal force and the ſquare of the periodic time conjointly.

3°. That in the ſame circle, the centripetal force, is as the ſquare of the periodic time inverſely.

4°. When the periodic time remains the ſame, the centripetal force is as the radius.

COR.

115. If a body revolves in ſuch a circle that the centripetal force be equal to that of gravity near the ſurface of the earth. Let d denote the ſpace fallen thro' in a ſecond, then $2d=\text{P}$, *Art.* 85, and $a\text{P}=cc$, gives $c=\sqrt{2ad}$; and $2ar=c\text{T}$, becomes $2ar=\text{T}\sqrt{2ad}$.

N. B. The velocity is expreſſed by the number of feet deſcribed uniformly in a ſecond and the time in ſeconds.

C O R.

116. If in CA produced to H, ſo that the centripetal force be equal to the force of gravity, at the diſtance CH (h), the force of gravity being unity, we have $P=\frac{hh}{yy}$, and by following the computation of *Art.* 112, we get $hh\times\frac{2a-y}{ay}=vv$, and $hh\times\frac{2a-d}{ad}=cc$, or becauſe $2ad-dd=bb$; we ſhall find $hh=cd\sqrt{a}$; and this value of cd wrote into $2arb=cdT$ gives $2ra^{\frac{3}{2}}=hT$, or $4rra^3=hhTT$. Conſequently, *the ſquares of the periodic times of bodies revolving in different ellipſes, about the ſame center or focus, are as the cubes of their mean diſtances.* For $4rr$ and hh remain here the ſame.

The famous aſtronomer *Kepler* was the firſt who diſcovered this law by obſervations; that the times were as the areas deſcribed by the radius drawn from the focus to the body, and that the orbit of all heavenly bodies were elliptical, and that the planets were attracted by the ſun, by forces that are as the ſquares of the diſtances inverſely.

C O R.

117. When $y=a$ in the ellipſis, $hh\times\frac{2a-y}{ay}=vv$, becomes $h=v\sqrt{a}$; and as $hh=cd\sqrt{a}$, *Art.* 116; when the ellipſis becomes a circle, we get $b=a$, and and $h=c\sqrt{a}$ in the circle. It follows, that the velocity of a body revolving in an ellipſis at the mean diſtance is equal to the velocity of a body deſcribing a circle at the ſame diſtance.

C O R.

118. Laſtly, from $P=\frac{hh}{yy}$, it appears that the centripetal

tripetal forces of bodies revolving in ellipſis at equal diſtances and about different centers, are as the ſquares *hh* of the diſtances CH, from the center C to the point H, where the centripetal force becomes equal to the force of gravity near the ſurface of the earth.

Before we can apply the foregoing theory to aſtronomy, it is neceſſary to know the mean diſtance of the earth from the ſun, and this diſtance can be obtained from no other means than by the ſun's parallax; but this, by all the care and induſtry of the aſtronomers, has not yet been diſcovered to any degree of exactneſs. But to aſcertain ſomething, we ſhall make uſe of the reſult of all the laſt obſervations on the tranſit of Venus, given in the Tranſactions, which makes that parallax to be 9.732″: and as the tangent of that angle is 47181,06744, the radius being 100000,00000,00000; then this tangent is to the radius as the ſemi-diameter of the earth (1) is to the diſtance required 21194. As the ſemi-diameter of the earth is 3959 Engliſh miles, that diſtance will be 83,907,046 miles. We ſhall however take 20000 ſemi-diameters for that diſtance in the following computations.

PERIODIC TIMES.

Saturn	Jupiter	Mars	Earth
10759.275,	4332.514,	686.9785,	365.2565,

Venus	Mercury	Moon
244.6176,	87.9692,	27.31.

Theſe times have been accurately obſerved by moſt aſtronomers, and are expreſſed in days and decimals.

Since the ſquares of the periodic times are as the cubes of the mean diſtances, *Art.* 116, we have

Mean Diſtances from the Sun to

Saturn	Jupiter	Mars	Earth	Venus	Mercury
190752,	104017,	30472,	20000,	14463,	7742.

Theſe diſtances are expreſſed in ſemi-diameters of the earth; and the diſtance of the moon from the earth has been found to be 60.11 of the ſame meaſure.

Diameters of the Planets.

Sun	Saturn	Jupiter	Mars	Earth	Venus
92.000,	7.260,	9.160,	0.523,	1.000,	1.033,

Mercury	Moon
0.368,	0.270.

Theſe diameters are proportional to their apparent ones, which have been determined by obſervation. Theſe bodies are nearly ſimilar, their contents are then as the cubes of their diameters.

Solid Contents of the

Sun	Saturn	Jupiter	Mars	Earth	Venus
778688,	383.76,	769.53,	0.1426,	1.000,	1.1023,

Mercury	Moon
0.510,	0.020.

Hence, the contents of the planets are in no proportion to their diſtances from the ſun, as might have been expected, if aſtronomers made no miſtake. The ſun is nearly a million of times bigger than the earth, and 670 times bigger than all the planets. Therefore the common center of gravity of the ſun and the planets muſt fall within the ſurface of the ſun; which accounts for the ſun's motion being ſo ſlow, in compariſon to that of the planets, as to appear to be at reſt.

We have found $2ra^{\frac{3}{2}} = hT$ in *Art.* 116. To adapt this equation to practice, it muſt be obſerved, that when the earth is ſuppoſed in the center of the motion, the diſtance is then equal to the ſemi diameter of the earth or unity, by ſuppoſition; and as the moon revolves about the earth in 27.32 days, and its mean diſtance

diſtance is 60.11; we have $a^{\frac{3}{2}} = 466.04$; which gives this proportion $27.32 : 466.04 :: hT ; a^{\frac{3}{2}}$, or $hT = 0.0586a^{\frac{3}{2}}$.

From whence we may compare the force of gravity at equal diſtances, or the quantity of matter, of ſuch bodies as have ſatellites, as follows:

If the ſun be in the center, then having the diſtance of the earth $a = 20000$, and the periodic time $T = 365.2565$; we get $h = 453.83$, or $hh = 205961.67$, for the quantity of matter in the ſun, when that of the earth is unity.

If Saturn be in the center, from Hugens's ſatellite, whoſe periodic time is 15.945 days, and its diſtance 135.65 from the center of Saturn, we get $h = 5.58$, or $hh = 31.14$. But if Jupiter be in the center, the periodic time of the fartheſt ſatellite is 16.6889 days, and diſtance 231.74; therefore $h = 12.25$ or $hh = 153.26$. Hence,

Quantity of Matter in the

Sun	Saturn	Jupiter	Earth
20596167,	3114,	15326,	100.

The quantity of matter in bodies that have no ſatellites, cannot be found by any method we know of.

As equal bodies placed on the ſurfaces of the planets, are attracted by forces, that are as the quantity of matter in the attracting body directly, and the ſquares of the diſtances inverſely; if then the laſt numbers are divided by the ſquares of the ſemi-diameters, we ſhall get for the

Attractions of equal Bodies placed on the Surface of the

Sun	Saturn	Jupiter	Earth
97335,	2363,	7306,	1000.

And if D expreſſes the denſity of a body placed in the center of attraction, then as the quantity of matter

ter is as the denſity and magnitude or cube of the radius conjointly: we have $hh = Dy^3$, or as $Pyy = hh$, by *Art.* 118, we get $P = Dy$. If therefore the laſt numbers are divided by their reſpective diameters, we get

	Denſity of the		
Sun	Saturn	Jupiter	Earth
2116,	657,	285,	1000.

Which ſhews that the denſity of the ſun is ſomething more than twice that of the earth, and thoſe of Saturn and Jupiter leſs.

We have given ſome of the principal applications of the general laws of motion laid down in this ſection, and which are particularly uſeful in aſtronomy; as to thoſe generally given, when the centripetal force is as any other power of the diſtances, have no foundation in nature, we have purpoſely omitted them, as a mere ſpeculation.

Problems of the Greateſt and Leaſt QUANTITIES.

LEMMA. *Fig.* 43.

119. *If from any two given points* D, K, *in a line* AC, *the indefinite lines* DE, KT, *are drawn at right angles to* AC; *and if a, m, are two conſtant quantities: It is required to draw lines from the given points* A, C, *to a point* L *in* DE, *ſo as* $a \times AL - m \times KN$ *ſhall be the leaſt poſſible.*

Let the angle DAE be ſuch that the radius is to its ſine, in the compound ratio of *a* to *m* and of CD to CK, then the point E will be the required one. From the point D and any other L in DE, the lines DF, LP, are drawn at right-angles to AE; then becauſe, 1°, CD : CK :: DL : KN, or, 2°, $a \times CD : m \times CK :: a \times DL : m \times KN$: and the right angled ſimilar triangles EPL, EFD, and DFA, give, 3°, DL : FP :: (LE : PE ::) AD : DF; but, 4°, AD : DF :: $a \times CD : m \times CK$ by conſtruction: therefore (No. 2, 3, 4)

4) DL : FP :: $a \times$ DL : $m \times$ KN, by equality of ratios, or $a \times$ FP $= m \times$ KN. Hence, $a \times$ AL $- m \times$ KN becomes equal to $a \times \overline{\text{AL} - \text{FP}}$, or if upon AE we take AM $=$ AL, then $a \times \overline{\text{AL} - m \times \text{KN}}$, or its equal $a \times \overline{\text{AL} - \text{FP}}$, becomes $a \times \overline{\text{AF} + \text{PM}}$: Which will be the leaſt of all when PM, vaniſhes, becauſe AF is conſtant; and this will happen when the point L falls upon the point E.

This lemma includes both M'Laurin's given in *Art.* 572 and 578; whereby all the problems called iſoperimetrical are ſolved in the moſt clear and elegant manner, without making uſe of any higher order of fluxions than the firſt.

C O R.

120. Hence, if CK $= d$, CD $= y$, DE $= x$, and AE $= z$, then DA : DF or $ay : md$ :: AE(z) : DE(x) or $axy = mdz$, when $a \times$ AL $- m \times$ KN is the leaſt poſſible. It muſt be obſerved that the radius DA muſt always be greater than the ſine DF, or ay greater than md, of the angle DAE, otherwiſe the problem would be impoſſible.

C O R.

121. If the point K coincides with the point D, or which is the ſame, when CD and CL become parallel, CK becomes equal to CD; that is, $y = d$, and KN $=$ DL; therefore $ax = mz$, when $a \times$ AE $- m \times$ DE is the leaſt poſſible, and this will be when the radius AD is to the ſine DF as a is to m or in a conſtant ratio.

C O R.

122. It is manifeſt, that when $a \times$ AL $- m \times$ KN is the leaſt poſſible $\frac{1}{m}$ AL $- \frac{1}{a}$ KN is likewiſe the leaſt poſſible, ſince both expreſſions give the ſame conſtruction; or from that either the greateſt or leaſt expreſſion, divided or multiplied by the ſame

ſame conſtant quantity *am*, does not change the queſtion.

C O R.

123. It is equally manifeſt, that when the firſt term $a \times AL$ or $\frac{1}{m}AL$ of the expreſſion is conſtant or given, the ſecond mKN or $\frac{1}{a}$KN will be the greateſt poſſible, when $a \times AL - m \times KN$, or $\frac{1}{m}AL - \frac{1}{a}KN$ is the leaſt poſſible. Since $m \times KN$ or $\frac{1}{a}KN$ may increaſe from o, but can never exceed $a \times AL$ or $\frac{1}{m}AL$.

C O R.

124. On the contrary, if the ſecond term $m \times KN$ or $\frac{1}{a}KN$ is conſtant or given, the firſt $a \times AL$ or $\frac{1}{m}AL$ will be the leaſt poſſible, when $a \times AL - m \times KN$ or $\frac{1}{m}AL - \frac{1}{a}KN$ is the leaſt poſſible. Since the firſt term may increaſe ever ſo much, but can never become leſs or equal to the ſecond.

PROBLEM. *Fig.* 39.

125. *To find the line* DMA, *drawn thro' two given points* D, A, *ſuch that a body ſhall deſcribe it in the ſhorteſt time poſſible, beginning to move from a ſtate of reſt, and being acted upon by the force of gravity only.*

Draw the horizontal line DB and AB perpendicular to it; and from any point M, the line MP parallel to DB: then if AB $= a$, AP $= y$, the arc DM $= z$, PM $= x$,

PM$=x$, c, a conſtant velocity, and v the velocity at M: Hence $\frac{\dot{z}}{v}$ expreſſes the fluxion of the time in deſcribing the arc DM by *Art.* 84, and $\frac{\dot{z}}{v}-\frac{\dot{x}}{c}$ the leaſt poſſible, when $v\dot{z}=c\dot{x}$, by *Art.* 121. Now, when the point M coincides with the loweſt point A, where $\dot{x}=\dot{z}$, we have $v=c$; ſo c denotes the velocity of the body at A: But the velocity at M is to the velocity at A, as the ſquare roots of AB and AP, by *Art.* 84, N°. 2, that is, $\surd\overline{a-y}:\surd\overline{a}::v:c$. Hence $\dot{z}\surd\overline{a-y}=\dot{x}\surd\overline{a}$, this ſquared and $\dot{z}^2-\dot{y}^2$ wrote for its equal $\dot{x}^2$, gives $a\dot{y}^2=y\dot{z}^2$ or $\dot{z}=\frac{a\dot{y}}{\surd\overline{ay}}$, whoſe fluent is $z=2\surd\overline{ay}+p$; which is the equation of the common cycloid by *Art.* 44; BD its baſe and AB the diameter of its generating circle, when $p=o$.

PROBLEM. *Fig.* 39.

126. *If the length of the curve* AMD *be given, or remains the ſame, to find its nature ſuch that the area* ABD *ſhall be the greateſt poſſible.*

If AM$=z$, PM$=y$, AP$=x$, and a a conſtant quantity; then will $a\dot{x}=y\dot{z}$, when $a\dot{z}-y\dot{x}$, is the leaſt poſſible, by *Art.* 121; and as the fluent of $a\dot{z}$ is given by ſuppoſition, that of $y\dot{x}$ of the area AMP, will be the greateſt poſſible, by *Art.* 123; but $a\dot{x}=y\dot{z}$ is the equation of the circle, by *Art.* 4. N°. 2; ſince the ſine y is to the radius a, as the fluxion $\dot{x}$ of the coſine is to the fluxion of the arc AM.

It is no leſs evident, that if the fluent of $y\dot{x}$, or the area ABD, is given or conſtant, the rectangle az or becauſe a is given, the arc z will be the leaſt poſſible, by *Art.* 124.

PROBLEM.

127. *If the ſurface deſcribed by the arc* AM, *about the axis* AP *be given; to find the nature of the curve ſuch, that the ſolid deſcribed by the area* AMP, *in the rotation, ſhall be the greateſt poſſible.*

It is evident, that when $ay\dot{z} - yy\dot{x}$ is the leaſt poſſible, we have $a\dot{x} = y\dot{z}$, by *Art.* 121; ſince the radius will be to the ſine of the angle made by the ordinate PM and the tangent at M as ay to yy, or as a to y. Conſequently the curve is again a circle. As the fluent of $ay\dot{z}$ denotes the ſurface deſcribed by the arc AM, when a is the circumference of a circle whoſe radius is unity, which being always the ſame, the fluent of $yy\dot{x}$ or the ſolid will be the greateſt poſſible, by *Art.* 123.

COR.

128. Hence, of all ſolids of the ſame boundary the ſphere is the greateſt, and of all equal ſolids the ſphere has the leaſt boundary. This has likewiſe been proved in *Art.* 32.

GENERAL COROLLARY.

129. If the radius be to the ſine of the angle at M, or which is the ſame, if $\dot{z}$ be to $\dot{x}$ as $Ay + Byy + Cy^3$, is to $a + by + cyv$, and $\dot{z} \times by a + by + cyy - \dot{x} \times by$, $Ay + Byy + Cy^3$, is the leaſt poſſible; we have $\dot{z} \times by Ay + By^2 + Cy^3 = \dot{x} \times by\ a + by + cy^2$, by *Art.* 121; therefore, if the fluent of $\dot{z} \times by, a + by + cyy$ be given, that of $\dot{x} \times by$, $Ay + byy + Cy^3$, will be the greateſt poſſible by *Art.* 123; and on the contrary, when the fluent of $\dot{x} \times by\ Ay + By^2 + Cy^3$ is given, that of $\dot{z} \times by$, $a + by + cyy$, will be the leaſt poſſible.

Obſerve that the coefficients a, b, c, A, B, C, are ſuppoſed conſtant but indetermined; and that any one or more of them may be unity or o.

To

To ſhew ſome particular example of this general expreſſion, let the length of the arc AM, and the ſurface deſcribed by it about AP be given: to find the nature of the curve, ſuch that the ſolid deſcribed by the area AMP in that rotation, ſhall be the greateſt poſſible: if we make $A=C=c=o$, and $B=1$, we ſhall have $\dot{z}yy=a\dot{x}+by\dot{x}$ for the equation of the curve required. And the ſum of the length of the arc AM, and the ſurface deſcribed by it, will be the leaſt poſſible, when the ſolid is given.

SECTION VI.

The MOTION *of* FLUIDS *and the* RESISTANCE *of* BODIES.

LEMMA. *Fig.* 44.

130. THE *force by which a fluid ſtrikes a plane in a perpendicular direction, is to the force by which it ſtrikes it in an oblique one, as the ſquare of the radius is to the ſquare of the ſine of the incident angle.*

With the radius CA deſcribe a quadrant DBA of a circle, draw the radius CB at pleaſure, and from the points D, B, the lines DL, BP, perpendicular to CB, CA; then if the radius CD expreſſes the abſolute force of the fluid in a perpendicular direction, DL will expreſs its effect againſt the inclined ſurface CB; or becauſe of the equal right-angled triangles CLD, CPB, DL is equal to CP, ſo that the abſolute force CD in a perpendicular direction, is to the relative force in an oblique direction CB, as CA is to CP: Now the number of particles ſtriking the oblique plane CB, at the ſame time as CA is CP: the abſolute force that ſtrikes a plane CB, in a perpendicular direction, is therefore to the part that ſtrikes it in an oblique direction, as the ſquare of the

radius CA is to the ſquare of the ſine CP of the incident angle DCL or CBP.

C O R.

131. Hence unequal planes differently inclined are acted upon by forces which are in the compound ratio of theſes planes and the ſquares of the ſines of the incident angles. For let P, Q, R, denote the forces which act upon the planes CA, CB, CE, reſpectively; then will $CA^2 : CP^2 :: P : Q$, by the laſt, and the forces acting upon inequal planes CB, CE, equally inclined, are as theſe planes, that is, $CB : CE :: Q : R$, and the compound of theſe two proportions gives $CB \times CA^2 : CE \times CP^2 :: P : R$.

P R O B L E M.

132. *If a machine be moved by means of a water-wheel, to find the velocity of that wheel, ſuch that the machine ſhall produce the greateſt effect poſſible.*

Let a denote the uniform velocity of the ſtream, x the velocity required: then will $a-x$ be the velocity by which the ſtream acts upon the wheel; and as the force of the ſtream is as the ſquare of the velocity, by *Art.* 131, $\overline{a-x}^2$, repreſents the force acting upon the wheel, and the velocity x multiplied by the force gives $x \times \overline{a-x}^2$ for the momentum of that force, by mechanics; which muſt be the greateſt poſſible: therefore $\dot{x} \times \overline{a-x}^2 - 2x\dot{x} \times a-x = 0$: hence, $x = \frac{1}{3}a$, which ſhews *that when the velocity of the wheel is one third of the velocity of the ſtream, the machine will produce the greateſt effect poſſible.*

C O R

133. Hence, if the value of x be wrote into the momentum $x \times \overline{a-x}^2$, we get $\frac{4}{27}a^3$ for the greateſt momentum poſſible; and $\overline{a-x}^2$ gives $\frac{4}{9}aa$, for the weight thereof in this caſe; that is, if the machine

chine be ſo loaded as to remain juſt at reſt, then by loading it by $\frac{4}{9}$ of that weight, it will produce the greateſt effect poſſible; ſetting aſide all friction or impediment.

LEMMA. *Fig.* 45.

134. *The ſines and coſines of any two arcs of a circle being given, to find the ſine and coſine of an arc equal to their ſum.*

Let BE, MP, be the ſines of the arcs BM, MA; BL the ſine of the ſum BMA, draw ED, EQ perpendicular to BL and CA: then becauſe the angles CEQ, DEB are equal the right-angled triangles CQE, BDE are ſimilar to the triangle CPM; and if the radius CM be unity, we have, 1°. CM : CP :: BE : BD = CP × BE, and CM : CE :: PM : QE = CE × PM. Now ſince the ſum of BD and QE is equal to the ſine BL, we have BL = CP × BE + CE × PM; that is, *the ſum of the rectangles made by the ſine of one arc and the coſine of the other, is equal to the ſine of the arc equal to their ſum.*

2°. CM : CP :: CE : CQ = CP × CE, and CM : PM :: BE : DE = PM × BE; and as the difference between CQ, DE, is equal to the coſine CL: we have CL = ± CP × CE ∓ PM × BE, according as the arc AB is leſs or greater than a quadrant. *Hence, the difference between the rectangles made by the coſines and by the ſines of any two arcs, is equal to the coſine of an arc equal to their ſum.*

PROBLEM. *Fig.* 46.

135. *The angles which the wind makes with both the axis and ſails of a wind mill, being given; to find the force with which the mill turns.*

Let CA be the direction of the axis, CQ that of the ſails, ſuppoſed to be flate, and CM that of the wind.

wind. From the center C deſcribe an arc AMF; draw MQ, AL perpendicular to CF and MP to CA: then becauſe the force of the wind againſt the ſails, being as the ſquare of the ſine QM of the incident angle FCM, by *Art.* 130; if we take CD equal to QM perpendicular to CQ, and draw DE perpendicular to CA; then will CD be to DE as the force of the wind to its part which turns the mill. Now, if the ſine PM$=x$, the ſine QM$=y$, and the radius unity, then will CP$=\sqrt{1-xx}$, CQ$=\sqrt{1-yy}$ and CL$=\sqrt{1-yy}\times\sqrt{1-xx-xy}$, by *Art.* 134, N°. 2: and becauſe CM is to QM, as the abſolute force of the wind is to its effect againſt the ſails, if the abſolute force be unity, we get y^2 for the force in the direction QM or DC; and becauſe of the ſimilar triangles ACL, CED, we have AC(1) : CL($\sqrt{1-yy}\times\sqrt{1-xx}-xy$) :: QM or CD($yy$) : DE$=y^2\sqrt{1-yy}\times\sqrt{1-xx}-xy^3$, for the force in the direction perpendicular to the axis CA.

C O R.

136. Since then the force by which a wind-mill is turned, depends upon the ſines of the angles made by the direction of the wind and poſition of the ſails in reſpect to the axis, it is therefore neceſſary to find what theſe angles ought to be, to produce the greateſt effect poſſible. But as there are two variable quantities, one muſt be ſuppoſed conſtant, and then the other; for conveniency we ſhall ſuppoſe firſt the ſine y of the angle MCQ conſtant. Now it is plain that the force $y^2\sqrt{1-yy}\sqrt{1-xx}-xy^3$, increaſes while x decreaſes; for $\sqrt{1-xx}$ increaſes and $-xy^3$ decreaſes; and therefore the force will be the greateſt poſſible when $x=o$, or when the direction CM of the wind coincides with the axis CA, let the ſails make what angle they will with the axis. This agrees with the common practice.

Now,

Now, the direction of the wind being ſuppoſed to coincide with the axis, the next inquiry is what angle the ſails are to make with the axis: when $x=0$, the expreſſion of the force becomes $y^2\sqrt{1-yy}$, which ought to be the greateſt poſſible; therefore its fluxion being made $=0$, gives $2y-2y^3=y^3$, or $y=\sqrt{\frac{2}{3}}$; and the angle correſponding to this ſine is 54°, 44′.

C O R.

137. If now, AC repreſents the direction of a ſhip, and CD the rudder, the reſt as before, we ſhall draw the ſame concluſions as we have done in regard to the wind-mill, *viz.*

1°. If the direction of the ſhip is in that of the current; the rudder CD ought to make an angle of 54°, 44′ with the ſhip, to make it turn round with the greateſt force poſſible.

2°. When the direction of the wind is in that of the ſhip, the ſails ought to be at right-angles to them. Since then the ſine QM (y) becomes equal to the radius CA; and its ſquare, which expreſſes the force, will be the greateſt poſſible.

3°. To find the angles made by the ſhip, wind and ſails; we ſhall* ſuppoſe y conſtant; and if s be the coſine of the angle QCM; then the force $yyx\sqrt{1-yy}+y^3\sqrt{1-xx}$ becomes $yyxs+y^3\sqrt{1-xx}$, whoſe fluxion being made $=0$, gives $s\sqrt{1-xx}=xy$, the ſquare of which is $ss-ssxx=y^2x^2$, or $ss=xx\times ss+yy$, or becauſe $ss+yy=1$, $ss=xx$ and $s=x$, or $s=-x$, which ſhews that the ſine of one angle muſt be equal to the coſine of the other.

This gives two different caſes; when the direction of the wind is between that of the ſhip and that of the ſails, as in this figure the angle ACQ muſt be a right one.

But

But when the direction CM of the wind is on one ſide, and the ſails CF on the other to that of the ſhip CA, as in *Fig.* 47; the ſine MP(x) becomes negative, and equal to the coſine of the angle FCA.

Now, as in theſe two different diſpoſitions one muſt be preferable to the other, we are then to find which of them, makes $yyx\sqrt{1-yy}+y^3\sqrt{1-xx}$ the greateſt.

In the firſt caſe, when the angle ACF, *Fig.* 46. is a right angle, the ſines x, y are both poſitive, and $x=\sqrt{1-yy}$, by the laſt, and hence the expreſſion of the force becomes y^2. In the ſecond x becomes negative, and that of the force is $-yyx\sqrt{1-xx}+y^3\sqrt{1-xx}$, which by ſubſtituting the former value x becomes $2y^4-y^2$. Hence, becauſe y muſt be leſs than unity, the firſt expreſſion y^2 of the force is greater than the ſecond $2y^4-y^2$; which ſhews that when the direction of the wind is between that of the ſhip and the ſails, and the angle ACF is a right one, the ſhip ſails with the greateſt velocity poſſible; and this caſe $yy+xx=1$, wherein y may be greater equal to, and leſs than x. And when $yy=\frac{2}{3}$, then will $xx=\frac{1}{3}$; and the angle FCA, 54°, 41′.

The ſhip cannot always ſail in the beſt manner; by making the fluxion of the force $2y^4-y^2$ equal to o, we get $\frac{1}{4}yy$ and ſo $xx=\frac{3}{4}$, gives the beſt poſition in this caſe.

The Chevalier *Renau* wrote firſt upon this ſubject in the year 1689; the famous *John Bernoullie* wrote againſt him in 1714, pretending that the Chevalier had been miſtaken in ſome caſes. But, according to what has been ſaid here, both of them ſeem to have been miſtaken in ſome caſes; however, their works are ſo full of mathematical learning as to deſerve being peruſed by the curious.

PROBLEM

PROBLEM. *Fig.* 48.

138. *Let a crooked tube* ABCD *of any diameter be filled with water to the height* LK; *to find the time and velocity of the deſcent and aſcent of the water.*

Let d denote the mean length of the tube filled with water, x the length of deſcent in one branch or aſcent in the other, v the velocity, and t the time; then will $\dot{x}=vi$, by *Art.* 81; and if m, n expreſs the ſines of the angles ABE, DCF of inclination of the branches, the radius being unity: then the whole length d is to x as the weight of the water (1) is to $\frac{x}{d}$ the weight in the direction of the branches, and the radius is to the ſine m or n as the weight $\frac{x}{d}$ in the direction of the branches to their effects $\frac{mx}{d}$, $\frac{nx}{d}$, in the vertical direction. Whence the force of acceleration, equal to the difference between the quantities of water in both branches, will be $\frac{m+n}{d}x$, and $\frac{m+n}{d}\times \dot{x}\dot{x}=v\dot{v}$, by *Art.* 82; whoſe fluent is $\overline{m+n}\times\overline{\mathrm{A}-xx}=dvv$. Now if x becomes a when $v=o$; then $\mathrm{A}=aa$, and if $aa-xx=yy$, the fluent becomes $\overline{m+n}\times yy=dvv$, or $y\surd\overline{m+}=nvd^{\frac{1}{2}}$, this value of v wrote into $\dot{x}=vi$, gives $\dot{x}d^{\frac{1}{2}}=yi\surd\overline{m+n}$. Whence if z denotes an arc of a circle whoſe radius is a and ſine y, we get $\dot{z}y=a\dot{x}$, by *Art.* 4. N°. 2, and ſo $\dot{z}d^{\frac{1}{2}}=ai\surd\overline{m+n}$, whoſe fluent is $zd^{\frac{1}{2}}=at\surd\overline{m+n}$; and when z denotes the ſemicircumference, the ratio of z to a is conſtant, and the time t that of the whole deſcent, which therefore is always the ſame, let that deſcent be what it will.

L COR.

COR.

139. Since the oſcillations of pendulums are in a ſubduplicate ratio to that of their lengths, by *Art.* 105, a pendulum, whoſe length is $\frac{d}{m+n}$, will perform its vibrations in the ſame time, that the water aſcends and deſcends in the tube.

PROBLEM. *Fig.* 49.

140. *The denſity of the air being every where as the force of compreſſion, to find its denſity at any given height form the ſurface of the earth.*

Let BMT be the logarithmical curve, AV its aſymptote, AB the ſurface of the earth, and PM any ordinate. If AB$=n$, PM$=y$, PV$=u$, and a a conſtant quantity or the ſubtangent; then will $a\dot{y}=y\dot{u}$, by the property of the curve; and ſince the denſity is proportional to the preſſure, $y\dot{u}$ ore it equal $a\dot{y}$, will expreſs the fluxion of the denſity, and therefore ay the denſity required.

COR.

141. If AP$=x$, then as the fluent of $a\dot{y}=y\dot{u}$, is $a\times\log.y=u$, by *Art.* 57; and when $y=n$, then $a\times\log.n=$AV; by ſubſtracting the firſt equality from the laſt we get $a\times\log.n-a\times\log.y=AV-PV=x$, or $x=a\log.\frac{n}{y}$. Therefore the denſity n at the ſurface being given, the height x which anſwers to any other denſity y may be determined.

COR.

142. Hence if d expreſſes the denſity of quickſilver, and b the height of a column of quickſilver equal to the weight of the atmoſphere, ſtanding upon the ſame baſe; then as an expreſſes the weight of a column of air at the ſurface, we have $an=dh$, and therefore,

therefore, the ſpecific gravity of the air and quickſilver being given together, with the height b of the mercury, the height of a column of air uniformly denſe with that at the ſurface, which is equal to the weight of the atmoſphere, will alſo be given.

C O R.

143 Hence, the weight of the atmoſphere increaſes or decreaſes as the mercury riſes or falls in the glaſs; which never riſes above thirty-one inches, nor falls lower than twenty-eight: theſe two numbers denote the extremes of the atmoſphere's weight. Mr. *Cotes* affirms the quickſilver to be 11900 heavier than air, or 14000 to 1; and Sir *Iſaac Newton* 11890; whence if we ſuppoſe $b=31$, $d=11900$ and $n=1$, we get $an=db=31\times 11900=368900$ inches; or $a=30742$ feet.

We have here conſidered gravity as uniform; but when the particles are attracted like other bodies, the problem will be different, in the manner following: Let C repreſent the center of the earth, $CA=r$ the radius, and the reſt as before: then will $\frac{-\dot{x}y}{r+x|^2}$ be the fluxion of the denſity or preſſure at P by ſuppoſition, and if in CP we take $\div CP : CA : CQ=z$; then $r+x=\frac{rr}{z}$, whoſe fluxion is $-\dot{x}=\frac{rr\dot{z}}{zz}$: Now the values of $\dot{x}$ and $r+x$ being wrote into the fluxion of the preſſure, gives $\frac{y\dot{z}}{rr}$. Whence, if the curve BNR be ſuch that the ordinate QN be always equal to the correſponding one, PM, the area CQNR divided by the ſquare of the radius r, will expreſs the preſſure at P. Becauſe the denſity PM at P is as the preſſure above it, its fluxion $\dot{y}$ is in a conſtant ratio to the fluxion $\frac{-\dot{x}y}{r+x|^2}$ or to its equal $\frac{y\dot{z}}{rr}$: and ſince $\dot{y}$ is to $\frac{y\dot{z}}{rr}$ as $\frac{\dot{y}}{y}$ to $\frac{\dot{z}}{rr}$

$\frac{\dot{z}}{rr}$ by diviſion; the logarithm of y will be to z in a given ratio; the curve BNR is therefore a logarithmic reverſed, and equal to BMT; and ſince AB is a common ordinate to them both, they muſt alſo have the ſame ſubtangent a: conſequently $a\dot{y}=y\dot{z}$, and $\frac{a\dot{y}}{rr}=\frac{y\dot{z}}{rr}$, whoſe fluent $\frac{ay}{rr}$ gives $\frac{an}{rr}$ for the preſſure at the ſurface A, and $a\dot{y}=y\dot{z}$ gives $aly=z$ and $aln=r$. Therefore $aln-aly$ or $al\frac{n}{y}=r-z$: but as $z=\frac{rr}{r+x}$, we get $al\frac{n}{y}=\frac{rx}{r+x}$: Now ſince x is but very ſmall in compariſon to the radius r of the earth, the attraction can cauſe no ſenſible error in the diſtance AP.

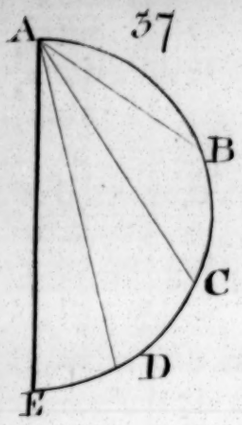
PROBLEM. *Fig.* 50.

144. *The denſity and elaſtic force of a medium being given; to find the velocity and time of a pulſe produced by a tremulous body in that medium.*

Suppoſe the medium placed in AC, to be moved from C towards A by the pulſe, whoſe extreme diſtance is AC, ſo that when the pulſe ceaſes, the elaſticity of the medium cauſes the particles of the medium to return to their former ſituations, and when the pulſe moves backwards, it will move the medium on the other ſide of the point C, at an equal diſtance CB to the former CA, and then ceaſes again; ſo that the pulſe will continually move the particles of the medium forwards from C to A, and backwards from C to B, as long as they continue: but while the medium placed in AC is moved from C to A by the pulſe, the medium placed in CB, will expand itſelf by its elaſtic force from B to A, and when the medium in BC is moved from C to B by the pulſe,

Plate IV.

37



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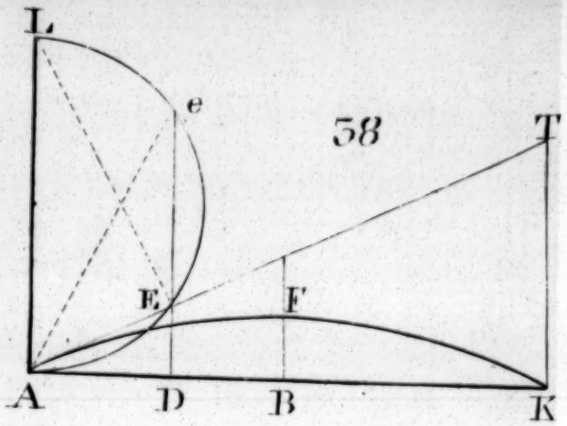

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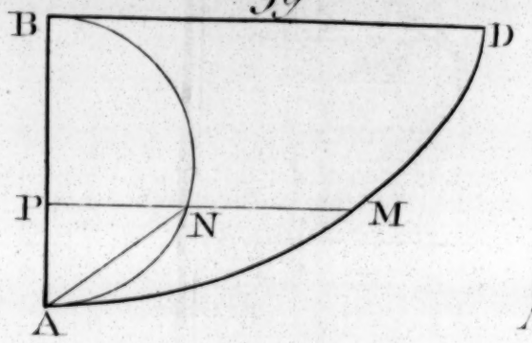

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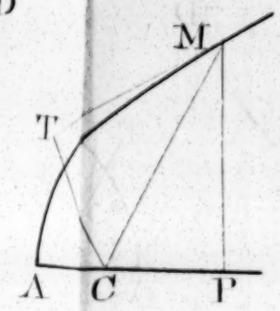

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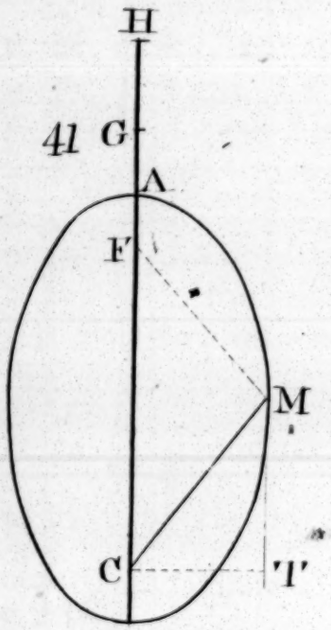

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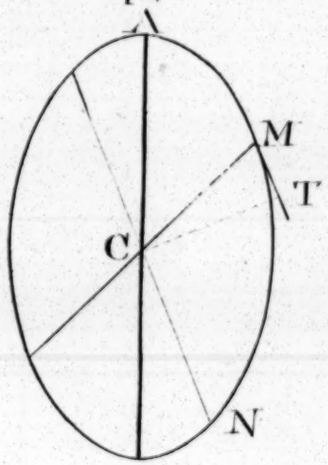

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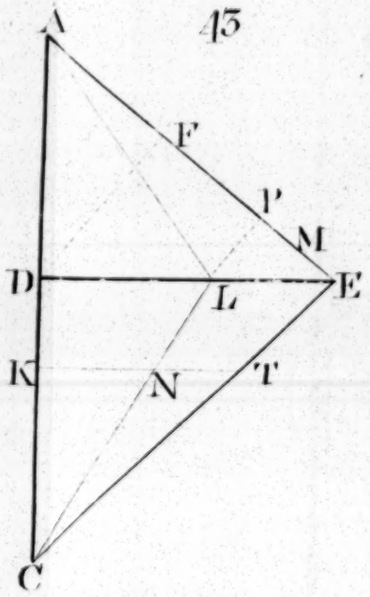

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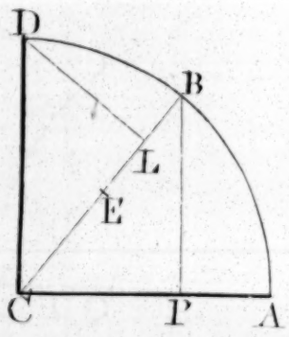

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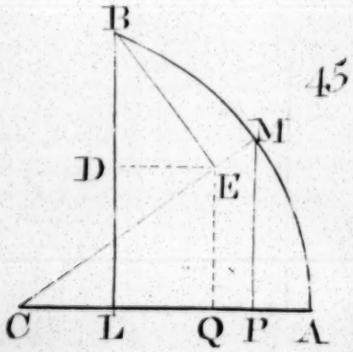

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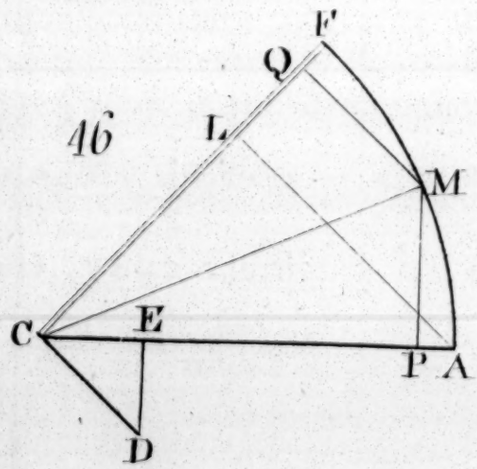

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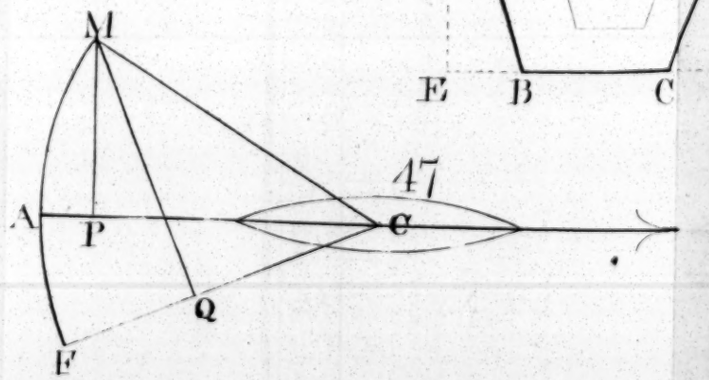

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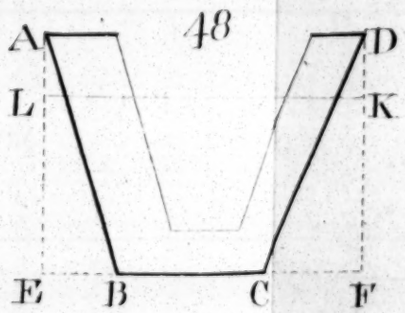

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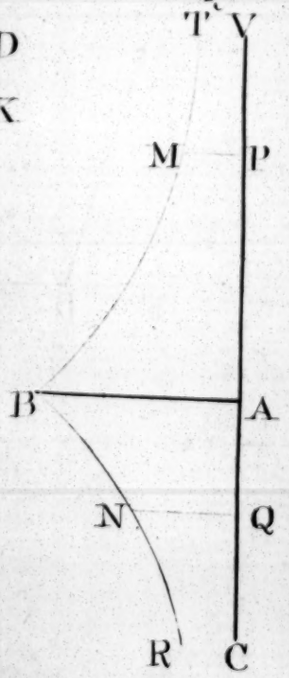

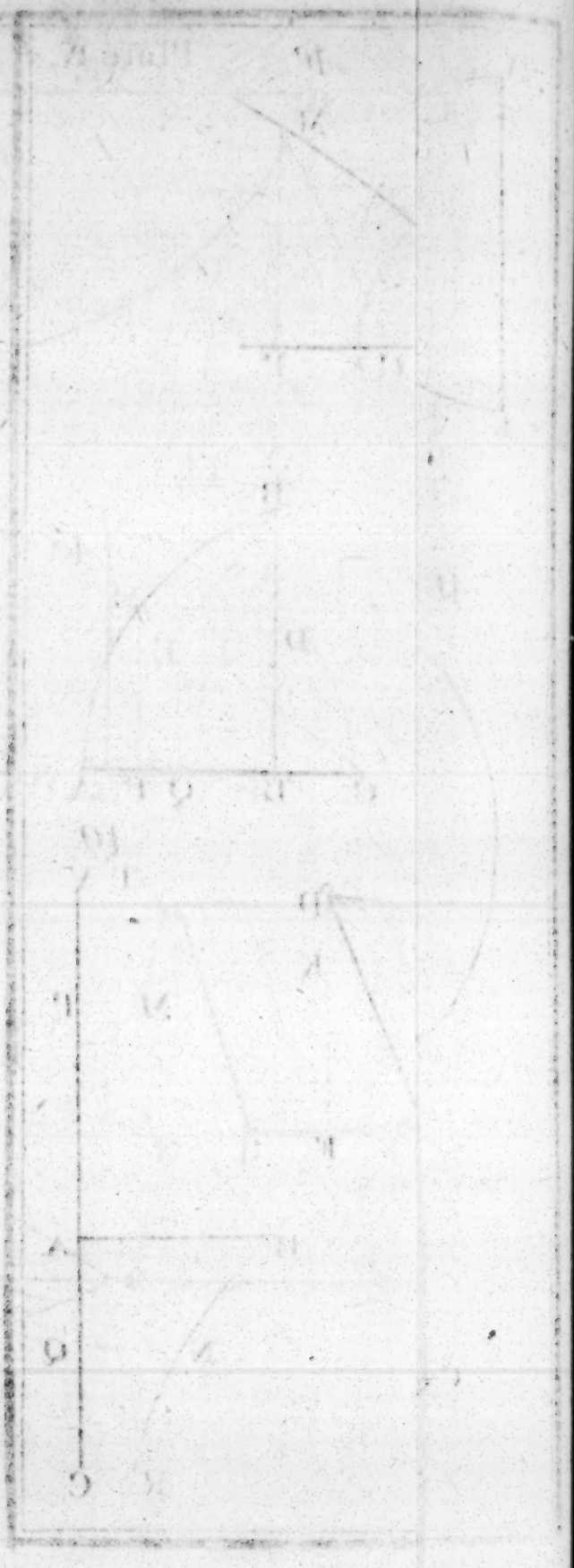
C

pulſe, that in AC expands itſelf from A to B by its elaſtic force: and for the ſame reaſon, when the medium in CA is moved by the pulſe from C to any intermediate point P, the medium in CB expands itſelf from B to P; the accelerative force in any point P is therefore as the difference between the forces of expanſions from A to P and from B to P; or which is the ſame, between the forces of the pulſe that moves the medium from C to P, and the elaſtic force which dilates the medium BC from B to P.

If A denotes the height of the medium of an uniform denſity, equal to that in which the pulſes are made, CA $=a$, CP $=x$, v the velocity, and t the time of motion from C to P; then as the elaſtic force is as the denſity and altitude of the medium by ſuppoſition, the rectangle aA will expreſs the elaſticity of the medium in its natural ſtate, and this force is increaſed in the motion from C to P reciprocally as AP the ſpace into which the medium AC is reduced; that is, the force at P in its return from A to P will be as $\frac{aA}{a-x}$, and for the ſame reaſon, the force of expanſion of the medium in BC, to BP, will be as $\frac{aA}{a+x}$; and their difference $\frac{aA}{a-x} - \frac{aA}{a+x}$ or $\frac{2axA}{aa-xx}$, will expreſs the accelerative force at P: but on account of the extreme ſhortneſs of the pulſes, xx may be neglected in the denominator without any ſenſible error, and therefore $\frac{2x}{a}$A expreſſes the accelerative force, which multiplied by $\dot{x}$, gives $-\frac{Ax\dot{x}2}{a} = v\dot{v}$, by *Art.* 82, whoſe fluent is B$-\frac{2Axx}{a} = vv$; but at

the

the point A, where $v=o$ and $x=a$, we get $B=2Aa$, and ſo $\frac{2A}{a}\times\overline{aa-xx}=vv$.

Now, if from the center C, a circle be deſcribed thro' the points A and B, and the ordinate PM drawn, and we make $PM=y$, the arc $AM=z$; then $yy=aa-xx$, and $\frac{2A}{a}yy=vv$, or $y\sqrt{\frac{2A}{a}}=v$, and this value of v wrote into $\dot{x}=vi$, *Art.* 81, gives $\dot{x}=iy\sqrt{\frac{2A}{a}}$, or becauſe $\dot{z}y=a\dot{x}$, by *Art.* 4, N°. 2, we get $\dot{z}=i\sqrt{2Aa}$, whoſe fluent is $z=t\sqrt{2aA}$; and when z becomes the ſemicircumference AMB, t expreſſes the time of a pulſe going from C to A and returning from A to C, which is therefore conſtant, let the extent of the pulſes be what it will.

C O R.

145. Since $\frac{2A}{a}yy=vv$; when $y=a$, at the point C, we get $2aA=vv$, or $\sqrt{2aA}=v$. Now, if b expreſſes the height thro' which a body falls in a ſecond then becauſe the times of falling bodies are as the ſquare roots of the ſpaces fallen thro', by *Art.* 84, N°. 2, we have $\sqrt{b}$ to $\sqrt{a}$, as one ſecond is to $\frac{\sqrt{a}}{b}$ the time fallen thro' a expreſſed in ſeconds; and the ſpace $\sqrt{2aA}$ divided by the time $\frac{\sqrt{a}}{b}$, gives $\sqrt{2bA}=v$ for the ſpace moved over uniformly in a ſecond, with the velocity v.

C O R.

146. It is manifeſt that velocity of pulſes, or the ſpaces uniformly deſcribed, in a ſecond, are as the ſquare root of the altitude A, in the ſame latitude; or the

the velocity v is alſo in a ſubduplicate of the elaſtic force directly and denſity inverſely, ſince Aa expreſſes the elaſtic force and a the denſity, by *Art.* 144.

147. The number of pulſes propagated, is always equal to the number of vibrations of the tremulous body and no more. For when the particle at C is moved by the pulſe from C to A, it will return again to C by its elaſtic force, where it will remain unleſs it receives a new motion either from the impulſe of the tremulous body, or from the pulſes propagated by the body. As ſoon, therefore, as the pulſes ceaſe to be propagated, the motion likewiſe ceaſes. It may be obſerved, that the velocity here determined is not abſolutely true, ſince in $\frac{2aA\dot{x}\dot{x}}{aa-xx}=-v\dot{v}$, we have neglected xx in the denominator; and the true fluent is $2aA\times by$ $\log.\overline{aa-xx}$, and $2aA\log.aa=vv$, or $\surd\overline{2aA\times\log.a}=v$; the value of v found before is however near enough in all caſes where the motion is not very intenſe or quick.

REMARK.

The foregoing principles may be applied to the motion of ſound, ſince it is excited by the tremulous pulſes of bodies, as experience ſhews. As the altitude A has been found to be 30742 feet, by *Art.* 143, and $b=16.1$, by *Art.* 107; therefore $32.2\times30742=989892.4$, whoſe ſquare root gives 994.93 feet for the diſtance the ſound moves over in a ſecond.

Sir *Iſaac Newton* finds 979 feet for that diſtance, becauſe he takes the altitude A ſomething leſs than we have done. But as this does not agree with the experiments, he adds 109 feet for the thickneſs of the particles of the air; and as this is not yet ſufficient, he ſuppoſed the motion of ſound increaſed in the ratio of 20 to 21, by the vapours floating in the air; then by means of theſe additions he finds that diſtance to be 1142 feet.

But

But as theſe ſuppoſitions are not grounded on any facts, I rather imagine that the altitude A has not been rightly determined, and for ſeveral reaſons.

1°. The denſity of the air is ſuppoſed to decreaſe from the ſurface upwards in the ratio of the ordinates of the logarithmic curve, but the particles near the ſurface being heated by the reflection of the ſun's rays, whereby they are more rarified than higher up.

2°. The air being loaded with vapours more or leſs, muſt needs vary that altitude.

3°. The air being ſo very light in compariſon to other bodies, it is extremely difficult to determine its denſity.

It appears therefore to me more expedient to determine the altitude A from the known velocity of the ſound, than this from the other, which may be done as follows: divide the ſquare 1304164 of 1142 feet by $2b$(32.2), which gives A=40502 feet for the true medium altitude of the air, or A=486024 inches.

And becauſe $an=dh$, by *Art.* 142, the altitude a(486024) of the atmoſphere is to the altitude h(31) of the quickſilver, ſo is the ſpecific gravity d(1400) of the ſpecific gravity n of the air, and therefore $n=\frac{43400}{486024}$, or $\frac{7}{8}$ nearly; whereas it is marked $\frac{5}{4}$ in the tables of Mr. *Cotes*, and generally ſuppoſed unity.

As the velocity of ſound has been determined by a great number of experiments, which were made by ſeveral eminent men, ſuch as Dr. *Hally* and *Derham*, and different times and diſtances, we have great reaſon to believe that this velocity has been much more accurately determined, than the ſpecific gravity by hydroſtatics, becauſe of the great difficulty there is in weighing ſo ſmall a volume of ſo very light a ſubſtance.

What confirms this determination beyond all exception, are the ſeveral accurate experiments made at *Minorca*,

Minorca, by General *Williamſon*, and ſeveral other officers of artillery, in order to determine the beſt length of guns; and we ſhall prove hereafter, in *Art.* 182, that the ſpecific gravity of air cannot be unity, much leſs greater, if theſe experiments can be depended upon, of which there is no doubt, becauſe I had it from moſt of theſe gentlemen, beſides the General himſelf.

PROBLEM. *Fig.* 51.

148. *The dimenſions of a cylindric veſſel*, ABCD, *conſtantly full of water, being given, as well as the diameter* LM *of the orifice in the bottom, to find the quantity of water that will run out of it in a given time.*

Let z expreſs the quantity of water in the veſſel, v the diſtance of its center of gravity from the bottom; then zv will expreſs its momentum againſt the orifice, by hydroſtatics, whoſe fluxion is $\dot{z}v+z\dot{v}$. Now if the height AB$=x$, the ſurface BC$=m$, that of the orifice LM$=n$; then will $mx=z$, by ſuppoſion, and the fluxion $m\dot{x}$ multiplied by the diſtance x gives $mx\dot{x}$ for the fluxion of the momentum likewiſe. Therefore $\dot{z}v+z\dot{v}=mx\dot{x}$. But when the ſurface BC deſcends to EF, it is plain that the quantity of water run out at the orifice muſt be equal to that contained in the ſpace EBCF; and becauſe the velocities of equal quantities of water thro' different openings, at the ſame time, are as theſe openings inverſely, by hydraulics: if $a+x$ denotes the height thro' which a falling body acquires the velocity of the water at the orifice, then will $mm:nn::$ $a+x:\frac{nn}{mm}\times\overline{a+x}=$ to the height thro' which a falling body would acquire the velocity of the water paſſing thro' the ſection BC. And ſince the momentum of the preſſure in any other ſection, is equal to

the difference of the momentums at BC and LM, the fluxion $\frac{nn\dot{x}}{mm}$ of the height $\frac{nn}{mm} \times \overline{a+x}$. Subſtracted from the fluxion $\dot{x}$ of $a+x$, gives $\dot{x}-\frac{nn\dot{x}}{mm}$ for the height thro' which a falling body would acquire the velocity of the deſcending water, and this multiplied by the ſection BC(m) gives $m\dot{x}-\frac{nn\dot{x}}{m}$ for the fluxion of the quantity run out in the time of the deſcent thro' BA, and ſince $\dot{z}$ expreſſes likewiſe the fluxion of that quantity, we have $\dot{z}=m\dot{x}-\frac{nn\dot{x}}{m}$; now this value of $\dot{z}$ and mx that of z found above being wrote into $\dot{z}v+z\dot{v}=mx\dot{x}$, gives $v\dot{x}-\frac{nn}{mm}v\dot{x}+x\dot{v}=x\dot{x}$, after having divided by m, or if we make $1-\frac{nn}{mm}=r$, we ſhall have $v\dot{x}r+\dot{v}x=x\dot{x}$.

To find the fluent of this equation, multiply both ſides by x^{r-1}, then will $rv\dot{x}x^{r-1}+\dot{v}x^{r}=\dot{x}x^{r}$, whoſe fluent is $vx^{r}=\frac{x^{r+1}}{1+r}$, or dividing by x^{r} and reſtoring the value of r, we get $v=\frac{mmx}{2mm-nn}$.

When the veſſel is not kept conſtantly full, that is, when it is not ſupplied by freſh water as it runs out, the deſcent of water will no more be uniform as before, but accelerated, and therefore, the height thro' which a falling body acquires the velocity, will be leſs at BC in proportion to that at the orifice; that is, we muſt from $2\dot{x}$ ſubſtract $\frac{nn}{mm}\dot{x}$, in order to get

$2\dot{x}-$

$2\dot{x} - \frac{nn}{mm}\dot{x}$ for the medium preſſure, and hence $2m\dot{x} - \frac{nn}{m}\dot{x} = \dot{z}$; by following the ſame ſteps as above, we ſhall have $v = \frac{mmx}{3mm - nn}$, in this caſe.

This is the demonſtration we promiſed in our elements of mathematics, page 273, it differs very little from what Sir *Iſaac Newton* ſays in his Principia, Book II. firſt Cor. after Prop. 36, but anſwers exactly his correction in that Prop. *Daniel Bernoullie* has treated this ſubject very copiouſly in his *Hydrodinamicæ*, but what he ſays in page 37 is quite a miſtake.

C O R.

149. Hence if KP, be perpendicular to the middle P, of the orifice, and equal to twice the height thro' which a falling body acquires the velocity of the water iſſuing thro' the hole, and the curve BL, CM, be ſuch that KI : KP :: PL^4 : IB^4; the velocity at I will be to the velocity at P, as KI is to KP: by *Art.* 559 of our Elem. and by the revolution of this curve about the axis KP, forms the cataract BLMC of the water, whoſe property is ſuch that its content expreſſes the quantity of water which preſſes upon the orifice in the ſame manner as if the reſt was congealed into ice. And therefore the other part of the water contained in the ſolid deſcribed by MCD, expreſſes the quantity which preſſes upon the ring deſcribed by MD, in the rotation.

PROBLEM. *Fig.* 52.

150. *If in the middle of the orifice* LM, *there was placed a ſmall circle ab, deſcribed about the center d, to find the quantity of water acb, ſupported by that circle.*

Imagine the figure 51 to revolve about CD, ſo that MD and CD, in this figure repreſent the radius *ad*, and altitude of the ſolid *acd*, ſupported by the ſmall circle *ab*; and call KI $=a$, IP $=x$, IC $=b$, PM $=y$, MD $=z$, : then will $y=b-z$; and by the property of the curve, we have $a:a+x::y^4:b^4$, or $\frac{ab^4}{y^4}=a+x$, or becauſe $y=b-z$, $x=\frac{ab^4}{\overline{b-z}^4}-a$, whoſe fluxion is $\dot{x}=\frac{4a\dot{z}b^4}{\overline{b-z}^5}$; therefore the fluxion of the ſolid deſcribed by MCD, will be as $\dot{x}zz$, that is as $\frac{4a\dot{z}z^2b^4}{\overline{b-z}^5}$, whoſe fluent is $\frac{abb}{3}\times\frac{4bz^3-z^4}{\overline{b-z}^4}$. Since the fluxion of this fluent is the ſame as the propoſed one. And as xzz expreſſes the cylinder of the ſame baſe and altitude as the ſolid *acb*; and $x=\frac{ab^4}{\overline{b-z}^4}-a$, that cylinder will be $azz\times\frac{b^4}{\overline{b-z}^4}-1$: conſequently the ſolid is to the cylinder as $\frac{abb}{3}\times\frac{4bz^3-z^4}{\overline{b-z}^4}$ to $azz\times\frac{b^4}{\overline{b-z}^4}-1$; or as $\frac{bb}{3}\times\frac{4bz^3-z^4}{zz\times\overline{b^4-\overline{b-z}^4}}$: but $b^4-\overline{b-z}^4=4b^3z^3-6bbz^4+4bz^5-z^6$.; and ſo the former proportion becomes that of $\frac{4}{3}b^3-\frac{1}{3}bbz$ to $4b^3-6bbz+4bz^2-z^3$, when divided by z^3.

Hence, if z becomes ſo ſmall in compariſon to b, that all the terms multiplied by that quantity may be neglected

glected, without any ſenſible error, the ſolid *acb* will be to the cylinder of the ſame baſe and altitude as $\frac{4}{3}b^3$ to $4b^3$, or as $\frac{1}{3}$ is to unity.

Sir *Iſaac Newton*, finds this laſt ratio to be as $\frac{1}{2}$ to unity, in *Cor.* 9. after *Prop.* 36. *Book* II. What induced him to think ſo (for he did not compute it as we have done) was, the ſolid *acb* muſt be leſs than a ſemi-ſpheroid, becauſe the angle *c* is acute, whereas in the ſpheroid it muſt be a right one, and the angle at *a* muſt alſo be acute; and as one half is an arithmetical mean between the cylinder and cone, he concludes, that this ſolid is one half of the cylinder. The reader may eaſily perceive the error of this concluſion; for tho' the former part of his argument, that this ſolid muſt be leſs than a ſemi-ſpheroid is true, he has by no means proved or attempted it, that this ſolid muſt be greater than the cone, whereas we have clearly proved that the ſolid and the cone muſt be equal, upon the ſuppoſition that *z* is incomparably leſs than *b*.

C O R.

151. Since we have $v=\frac{mmx}{2mm-nn}$, by *Art.* 148. when the orifice LM becomes ſo great as to be ſenſibly equal to BC, which will happen when the circle *ab* is placed in an unconfined fluid: then *n* may be taken equal to *m*; and hence the equality above becomes $v=x$; that is the velocity of the fluid is equal to that which a body acquires by falling thro' the height *cd* of the fluid.

C O R.

152. Hence, if the water was at reſt, and the little circle *ab*, did aſcend, with the ſame velocity with which the water deſcended thro' the orifice, or that acquired by a body falling thro' the height *cd*; the reſiſtance of the little circle *ab* will be equal to the weight

weight it ſupports when at reſt, and the water deſcends. Therefore the reſiſtance of the little circle in the aſcent will be equal to one third of the weight of the cylinder of the ſame baſe and altitude to *cd*.

C O R.

153. Since the ſolid *acb* is one third of the cylinder of the ſame baſe and altitude, by *Art.* 150. the circle *ab* muſt move with triple the velocity of the water in its deſcent, at the ſame time, in order to meet with a reſiſtance equal to the weight of the cylinder of the ſame baſe and altitude *cd*: and ſince the heights of falling bodies are as the ſquares of the velocities acquired in the fall, by *Art.* 84. No. 2. the height thro' which a falling body acquires a velocity triple that of another, muſt be nine times greater. Conſequently a body muſt fall thro' a ſpace nine times the height *cd*, to acquire the velocity with which the circle *ab* muſt aſcend, in order to meet with a reſiſtance equal to the weight of a cylinder of water of the ſame baſe *ab*, and altitude *cd*. Becauſe the reſiſtance of a cylinder, moving in the direction of its axis, is the ſame as that of its baſe: a cylinder whoſe baſe is equal to the little circle *ab*, moving in the ſame direction and with the ſame velocity, will meet with the ſame reſiſtance as that circle.

If the length of the cylinder, moving in a fluid, be increaſed or diminiſhed, its motion, as well as the time of deſctibing nine times its length, will be increaſed or diminiſhed in the ſame ratio; and therefore the force by which the motion ſo increaſed or diminiſhed, can be deſtroyed or generated, will continue the ſame; becauſe the time is increaſed or diminiſhed in the ſame proportion; and conſequenty the force remains ſtill equal to the reſiſtance of the cylinder, ſince that reſiſtance will alſo remain the ſame.

If

If the denſity of the cylinder be increaſed or diminiſhed, its motion, as well as the force by which its motion may be generated ordeſtroyed in the ſame time, will be increaſed or diminiſhed, in the ſame ratio. The reſiſtance of any cylinder whatever, will conſequently be to the force by which its whole motion may be generated or deſtroyed, in the time it moves over nine times its length as the denſity of the medium is to the denſity of the cylinder nearly.

Sir *Iſaac Newton* obſerves, that a fluid muſt be compreſſed to become continued; it muſt be continued and non-elaſtic, that all the preſſure ariſing from its compreſſion may be propagated in an inſtant, and ſo acting equally upon all the parts of the body moved, may produce no change in the reſiſtance. The preſſure ariſing from the motion of the body, is ſpent in generating a motion in the parts of the fluid, and this creates the reſiſtance. But the lateral preſſure ariſing from the compreſſion of the fluid, be it ever ſo forcible, if it is propagated in an inſtant, generates no motion in the parts of a continued fluid, nor produces no increaſe or decreaſe in the reſiſtance.

LEMMA.

154. *The reſiſtance of a body moving in a compreſſed non-elaſtic fluid, will always be the ſame, let its figure be what it will, when the greateſt tranſverſe ſection is the ſame.*

For whether *ab* be the diameter of a circle, or the ſection of any ſurface, the ſolid *acb* of the fluid ſupported by that ſurface, will be always one third part of the cylinder of the ſame baſe or altitude as has been proved in *Art.* 150. Conſequently the reſiſtance of any ſolid is always equal to that of its tranſverſe ſection.

N. B.

N. B. We ſuppoſe the friction the ſolid meets with in its motion to be inconſiderable, and the particles of the fluid to have no coheſion.

THEOREM.

155. *If a globe moves uniformly forward, in an infinite compreſſed non-elaſtic medium, its reſiſtance will be to the force that may generate or deſtroy its whole motion in the time it moves over the length of ſix of its diameters, as the denſity of the fluid is to the denſity of the globe.*

For ſince the reſiſtance of the globe is equal to that of its circumſcribed cylinder, and the reſiſtance of the cylinder is to the force that may generate or deſtroy its whole motion, in the time it deſcribes nine diameters, as the denſity of the medium is to the denſity of the cylinder, by *Art.* 153. Therefore as the globe is the two thirds of the cylinder, the reſiſtance of the globe is to the force that may generate or deſtroy its whole motion, in the time it deſcribes ſix of its diameters, as the denſity of the medium is to the denſity of the globe.

COR.

156. Hence the reſiſtances of globes, in an infinite compreſſed non-elaſtic mediums, are in the compound ratio of the ſquares of their velocities, of the ſquares of their diameters and the denſities of the mediums directly, and the denſities of the globes inverſely. For the reſiſtance increaſes in proportion as the ſquares of the velocities increaſe, it increaſes alſo as the ſurface of the globe, or the ſquare of its diameter increaſes, as likewiſe as the denſity of the medium increaſes, and decreaſes as the denſity of the globe increaſes.

COR.

COR.

157. If different globes move in the ſame medium, and with the ſame velocity, their reſiſtances will be as the ſquares of their diameters directly, and their weights inverſely; and if the globes are of the ſame denſity, their reſiſtances are as the ſquares of their diameters directly, and the cubes of the ſame diameters inverſely; that is as their diameters inverſely.

Thus the reſiſtance of a nine pound iron ball, whoſe diameter is four inches, is to the reſiſtance of an iron ball, whoſe diameter is ſeven inches, as ſeven to four.

COR.

158. The greateſt velocity a globe can acquire in a reſiſting medium by its comparative weight, is equal to that which it may acquire by falling with the ſame weight, from a ſtate of reſt in a non-reſiſting medium, thro' a ſpace that is to three of its diameters as the denſity of the globe is to the denſity of the medium. For the globe, in the time of its fall, will acquire a velocity, which, being uniformly continued, will deſcribe a ſpace double that fallen thro' in that time. And therefore the reſiſtance will then be equal to its gravity, by *Art.* 153: and after this the motion can no more accelerate and become uniform.

COR.

159. Hence if d denotes the diameter of the globe, m its denſity, n that of the medium, and r twice the height from which a falling body acquires the greateſt velocity; then will $r : 6d :: m : n$, or $r = \frac{6md}{n}$, by *Art.* 158.

REMARK I.

Sir *Iſaac Newton* found $r = \frac{8md}{3n}$, in the 2d *Cor.* after *Prop.* 38, Book II. in conſequence of his finding the weight ſupported by a ſmall circle placed in the middle of an orifice, to be one half; whereas we proved it to be but one third of the cylinder, which has the ſame baſe, and whoſe altitude is equal to the height thro' which a falling body may acquire the velocity of the water. Now as the laws of motion in a reſiſting medium abſolutely depend on the determination of the greateſt velocity of bodies, it will be ſhewn hereafter, in *Art.* 179, that our determination of this velocity is confirmed by ſome undoubted experiments, and that Sir *Iſaac*'s cannot be true.

REMARK II.

Sir *Iſaac Newton* conſiders the air to be an infinite compreſſed fluid of the ſame claſs as water, which does not ſeem to agree with its compreſſibility and elaſticity, but rather as a rare elaſtic medium. In this caſe, a globe moving forward uniformly meets with a reſiſtance that is to the force by which its whole motion may be generated or deſtroyed, in the time it deſcribes two thirds of its diameter, as the denſity of the medium is to the denſity of the globe. This Sir *Iſaac* has demonſtrated in *Prop.* 35, *Book* II. and I in my Artillery, *Theor.* 5, of the Introduction. But it is ſuppoſed, in this laſt cited propoſition, that the particles of the medium vaniſh immediately after the ſtroke, ſo as not to prevent the next from ſtriking the body in the ſame manner: this certainly is not the caſe in the air, which compreſſes in a heap before the body, and forms a kind of pointed

ed ſolid, and if it is nearly the ſame as we have ſhewn it does in water, the globe muſt then move over nine times the two-thirds of its diameter, that is ſix of its diameters, by *Art.* 150, to meet with a reſiſtance that is to the force which may generate or deſtroy its whole motion, in the ſame time as the denſity of the medium is to the denſity of the globe; and conſequently the ſame as above. Hence, whether the air be conſidered as an infinite compreſſed fluid, or as a rare elaſtic medium, the concluſion will be the ſame.

SECTION VII.

Of MOTION *in a* RESISTING MEDIUM.

THE laws of motion conſidered in a non-reſiſting medium, and which we have treated of in the laſt ſection, have formerly been looked upon as univerſal, and were applied to all caſes in general, till Sir *Iſaac Newton* found that they could only hold good in thoſe bodies which moved at a great diſtance from the earth, ſuch as the planets and their ſatellites, which are ſuppoſed to deſcribe their orbits in a free ſpace, void of all reſiſtance: it is true, that no abſolute vacuum can be conceived, yet as the medium in which the heavenly bodies move, muſt be infinitely rare, ſo as to cauſe no retardation perceivable in many ages, it may be neglected. But in the motion of bodies near the ſurface of the earth, the caſe is quite different; ſince the particles of air reſiſt the bodies conſiderably, and therefore all that had been ſaid before of projectile motions is erroneous and uſeleſs.

To diſcover the laws by which bodies are reſiſted, Sir *Iſaac* made a great many experiments with pendu-

ums, moving in the air and in water; he did likewiſe let fall ſeveral globes of different denſities from the cupola of St. *Paul*'s, and compared the times of their fall with the times they would fall in a non-reſiſting medium, by which he was able to compute their reſiſtance. It is true that theſe balls were too ſmall, and the diſtances fallen thro' too little, to determine the reſiſtance in a very exact manner, yet they were ſufficient to confirm the theory he had eſtabliſhed before hand, in relation to the motion when the bodies were reſiſted by a medium.

In the theory of powder, we are obliged to ſuppoſe that when it is fired, it acts in the ſame manner as an elaſtic fluid, becauſe we have no other principles to go by; but as powder does not light inſtantaneouſly, and we do not know by what law the inflammation proceeds, it is evident it cannot act with the ſame force as an elaſtic fluid of the ſame denſity, and conſequently, all the concluſions drawn from the ſuppoſition, are uncertain and erroneous.

It has hitherto been ſuppoſed that the more powder a piece was loaded with, the longer it ſhould be, in order to give time for the powder to light; and it ſeems to be reaſonable that it ſhould be ſo, but notwithſtanding experiments have proved the contrary; for if a piece be loaded with various charges, it will be found that there is one which will produce a greater effect than any other that is more or leſs: again, if ſeveral pieces of the ſame caliber, and of different lengths, are tried, it will be found that there is one certain length which produces a greater effect than any other.

It has likewiſe been found by experiment, that a mortar loaded with a quantity of powder, which weighs but one ſixth part, the weight of its ſhell when filled, carried its ſhell farther than a gun loaded ever

ever ſo much, when both are elevated at an angle of the ſame degrees: this alone is ſufficient to ſhew our ignorance in regard to the force of powder; for as the bore of a mortar is conſiderably wider than the chamber, it ſeems that as ſoon as the ſhell has moved from its place, the powder can act but very little upon it, and therefore the length of a piece ſeems to contribute but little to increaſe the force of powder; yet in experiments made with guns, it is found that there is a certain length, which is neceſſary to produce the greateſt effect. This will be proved hereafter, when we treat of projectile curves deſcribed in the air, and confirmed by unexceptionable experiments.

Mr. *Robins*, in his treatiſe of gunnery, tried by a great many experiments to determine the force of powder, but finds the velocities in general much greater than they poſſibly can be; which ſeems to be owing to the very ſmall quantities of powder that he was able to uſe, for the leaſt error or overſight in ſo ſmall a quantity, muſt neceſſarily become very conſiderable in ſuch large charges as are uſed in pieces of artillery. Theſe experiments were intended to confirm his theory, which, in itſelf, was built upon erroneous principles, as we have ſhewn in our treatiſe of artillery; and it required no ſmall degree of dexterity to confirm a theory built upon wrong principles, by experiments no leſs faulty in their contrivance.

The learned profeſſor *Euler*, in his Comment upon *Robins*'s work, page 566, gives a table of velocities, produced by pieces of different lengths, and loaded with various quantities of powder; wherein he committed the ſame error as his author, by making them much greater than they poſſibly can be; beſides he ſuppoſes that all cannon of the ſame number of calibers long, and loaded proportionally, will carry their

ſhots

ſhots equally far, which we have proved not to be true; neither does he ſeem to take notice of Sir *Iſaac*'s theorem, where he ſhews how to find the greateſt velocities; nor has he given any experiments to confirm his aſſertions. Thus, for inſtance, when the piece is twenty-two calibers long, and loaded with half the weight of the ſhot he finds 1477 Rhinland feet, or 1521 *Engliſh*, moved over in a ſecond; whereas we prove that this velocity can never amount to 914.7 feet in a forty-two pounder, and conſequently much leſs in a ſmaller caliber.

He gives likewiſe another table in page 600, which contains the greateſt and beſt charges of pieces, whoſe lengths are expreſſed in calibers; thus when the piece is twenty-two calibers long, he finds the greateſt charge to be 5.96 calibers in length; now as in our guns, four calibers in length contain an equal weight to that of the ſhot, therefore the beſt charge of an eighteen pounder is 26.82 lb. According to this table, that of a twenty-four pounder 33.96 lb and that of a forty-two, is 62.58 lb. Now as it has been found by experiment, that nine pound of powder in an eighteen pounder of about this length was the beſt charge; and therefore 12 lb. for a twenty-four pounder, and 21 lb. for a forty-two pounder: the charges given in this table are above double thoſe they ought to have. Beſides if theſe charges were true, they could never be put in practice, ſince theſe pieces would become twice heavier than they are now to reſiſt ſo great a force as would be produced by theſe charges. As to the charges given to battering pieces, it has been found by experiments, that one third of the weight of the ſhot is ſufficient; in all other occaſions, a leſs quantity will do.

In

In this ſection, after having given ſome general theorems, and a general problem of projectile curves, we give ſeveral examples of bodies moving in right lines, from whence we deduce ſome uſeful remarks relating to artillery, in order to determine the beſt length and charge of fire arms; this being done we treat of the curve deſcribed in the air, which is the moſt difficult part of this work. As no author has yet determined this curve in all caſes, we have been ſo much the more attentive, and given various numeral examples, by three different methods, in ſuch a manner as to leave no doubt or obſcurity, to prevent any miſunderſtanding of the ſubject.

THEOREM. *Fig.* 53.

100. *The fluxion of the velocity generated by any power in a reſiſting medium, is as the rectangle made by the difference between the meaſure of the power and the reſiſtance, and the fluxion of the time elapſed from the beginning of the motion:*

Let the ordinate PM, of the curve AM conſtantly expreſs the meaſure of the power, and the ordinate PN of another curve AN the correſponding reſiſtance of the body; the abſciſſa AP expreſſing the time from the beginning of the motion: it is evident, that the difference MN, expreſſes the force which generates the velocity: for the area AMP, expreſſes the ſum of all the powers from the beginning of the motion, and the area ANP the ſum of all the reſiſtances, and therefore the difference NAM between theſe areas the velocity at the end of the time AP: but the fluxion of this area is equal to the rectangle made by NM and the fluxion of AP, by *Art.* 16; conſequently the fluxion of the velocity is equal to the rectangle made by the difference NM, between the meaſure of the power and the reſiſtance and the fluxion of the time elapſed.

THEO.

THEOREM.

161. *The fluxion of the ſpace deſcribed in a reſiſting medium, is as the rectangle made by the velocity and the fluxion of the time elapſed from the beginning of the motion.*

For ſince the ſpace MAN is as the velocity, and the fluxion of the ſpace deſcribed as the rectangle made by the velocity and the fluxion of the time, by *Art.* 81; the reſiſtance having no other effect than to diminiſh the velocity, the demonſtration is here the ſame as in that article.

THEOREM.

162. *The reſiſtance acts only in the direction of the moving body.*

For in all other directions the body is preſſed or acted upon, equally and in oppoſite directions; the reſiſtance can therefore affect the motion no where but in the direction of the body in motion.

GENERAL PROBLEM. *Fig.* 54.

163. *If a body, projected from a given point* A *in a direction* AE, *with a given velocity, be acted upon by any centripetal force, in a reſiſting medium; to find the nature of the curve* AM *deſcribed by the body.*

Let CM $=y$, MT $=\dot{z}$, the perpendicular TR to CM, $\dot{x}$, RM $=\dot{y}$, the perpendicular CS to MT, s; p, the centripetal force; R the reſiſtance; v, the velocity, and t the time: then will $\dot{z}=vi$, *Art.* 161; and as $\frac{\dot{y}}{\dot{z}}p$ expreſſes the part of the centripetal force p which acts in the direction of the tangent, we have $\dot{v}=\pm\frac{\dot{y}}{\dot{z}}p-\mathrm{R}\times by\dot{t}$, by *Art.* 160;

or

or writing the value of $\dot{t}$, it becomes $v\dot{v} = \pm p\dot{y} - R\dot{z}$; the ſign $+p\dot{y}$ prevails in the deſcent, and $-p\dot{y}$ in the aſcent. Conſequently the laws of the centripetal force p and of the reſiſtance R being given, the nature of the curve AM may be determined.

C O R.

164. When the directions of the centripetal force p are parallel, the laſt equation remains the ſame, only $\dot{y}$ expreſſes the fluxion of the ordinate and $\dot{x}$ that of the abſciſſa; we have beſides $p\dot{z}^2 + v\dot{v}\ddot{y} = o$, by *Art.* 92. when $\dot{x}$ is conſtant, and $p\dot{x}\dot{y} = v\dot{v}\ddot{x}$ when $\dot{z}$ is conſtant; ſince the reſiſtance of the medium does not affect the equation, which does ariſe from the direction perpendicular to the tangent, by *Art.* 162.

C O R.

165. If r denotes twice the height from which a falling body acquires a velocity equal to the greateſt which that body can acquire in the reſiſting medium, we have $R = \frac{v^2}{r}$; for when v becomes the greateſt velocity or $r = v^2$, the reſiſtance becomes equal to the force of gravity, as has been ſhewn before in *Art.* 158; and after that the motion becomes uniform in the deſcent, or diminiſhes in any other direction.

EXAMPLE.

166. Let a body deſcend from a vertical height by its own gravity, in a medium that reſiſts according to the ſquares of the velocities; then as $z = y$ in this caſe, and $R = \frac{v^2}{r}$, the equation $v\dot{v} = \pm p\dot{y} - R\dot{z}$ becomes $v\dot{v} = \dot{y} - \frac{v\dot{v}}{r}y$, or $\dot{y} = \frac{rv\dot{v}}{r - v^2}$, ſuppoſing gravity to be unity: whoſe fluent is $y = A - \frac{1}{2}r$ log.

$\overline{r-vv}$; but when $y=o$, then will alſo be $v=o$; and therefore $A=\frac{1}{2}r$ log. r; conſequently the compleat fluent is $y=\frac{1}{2}r$ log. $\frac{r+o}{r-vv}$. Or if q expreſſes the number whoſe hyperbolic logarithm is $\frac{2y}{r}$; then will $q=\frac{r+o}{r-vv}$, from whence we get $vv=\overline{q-1}\times\frac{r}{q}$.

Becauſe $\dot{y}=\dot{z}$, $\dot{z}=v\dot{v}$, becomes $\dot{y}=v\dot{v}$, and ſince $\dot{y}=\frac{rv\dot{v}}{r-vv}$, we get $\dot{t}=\frac{r\dot{v}}{r-vv}$. If we ſuppoſe a the greateſt velocity, that is $r=aa$, the fluent of this fluxion will be $t=\frac{1}{2}a$ log. $\frac{a+v}{a-v}$; conſequently the height y fallen through being given, the velocity v will be given, and from the known velocity the time t may be found.

APPLICATION.

Let $y=1848$, and the diameter of an iron ball four inches; then if $n=1$ we have $m=7425$, by our tables, and $r=\frac{6md}{p}$ (*Art.* 159.) becomes $r=14850$ feet. Hence $\frac{2y}{r}=.249$ nearly, this being the hyperbolic logarithm of the number q; and to find its number, we muſt find firſt its tabular logarithm, which is done by multiplying the hyperbolic logarithm .249 by the conſtant number .43429, by *Art.* 59; therefore that logarithm is 108138, and the number anſwering in the common tables is $q=1.282746$; hence $v=\overline{q-1}\times\frac{r}{q}=3273.27$; this number multiplied by 32.2, twice the height fallen thro'

thro' in a ſecond, by *Art.* 86. gives 105399.294, whoſe ſquare root 324.65 will be the number of feet moved over uniformly in a ſecond with the velocity acquired in the deſcent thro' 1848 feet. The deſcent thro' the ſame height in a non-reſiſting medium would be 345 feet.

To find the time, by means of the equation, $t = \frac{1}{2}a \log.\frac{a+v}{a-v}$, we have $a = 121.86$, $v = 57.21$, hence $a+v = 179.07$, $a-v = 64.65$, and $a+v$, divided by $a-v$, gives 2.7698, whoſe tabular logarithm is .442448; which being multiplied by the conſtant number 2.302585, gives 1.01877 for the hyperbolic logarithm of that number, by *Art.* 58; and this number multiplied by $\frac{1}{2}a = 60.93$, gives $t = 62.0736$ feet, which divided by 5.67, the ſquare root of 32.2, (*Art.* 85.) gives 10.94 ſeconds.

In a non-reſiſting medium the body would have fallen thro' the ſame height in the time of 10.71 ſeconds, and therefore the difference is 0.23 ſeconds only.

EXAMPLE.

167. The body being projected upwards in a vertical direction, with a given velocity c; to find the height to which the body will riſe, as well as the velocity and time. We have $v\dot{v} = -\dot{y} - \frac{vv}{r}\dot{y}$ in this caſe, when gravity is unity, or $\dot{y} = \frac{-rv\dot{v}}{r+vv}$, whoſe fluent is $y = A - \frac{1}{2}r \log.\overline{r+vv}$; but when $v = c$, then $y = o$; and $A = \frac{1}{2}r \log.\overline{r+cc}$; therefore $\frac{2y}{r} = \log.\frac{r+cc}{r+vv}$. If q expreſſes the number whoſe hyperbolic logarithm is $\frac{2y}{r}$, then $q = \frac{r+cc}{r+vv}$; from whence we get $vv = \frac{r+cc}{q} - r$.

The equation of the time will be $\dot{t}=\frac{-r\dot{v}}{r+vv}$; hence if A denotes an arc of a circle, whoſe radius is $r^{\frac{1}{2}}$, and tangent c and n another arc of the ſame radius, but v the tangent, we ſhall have $t=A-n$.

APPLICATION.

Let $cc=5280$ feet and the reſt as before; then when the body arrives to its higheſt point, we have $y=\frac{1}{2}r$ log. $\frac{r+c}{r}=\frac{1}{2}r$ log. 1.3555; the tabular logarithm of this number is .1311, which multiplied by the conſtant number 2.30258, by *Art.* 59, gives .30187 for the hyperbolic logarithm of that number, and this logarithm multiplied by $\frac{1}{2}r$, 7425, gives $y=2241$ feet. As the body would have riſen to the height of $\frac{1}{2}cc$ or 2640 feet in a non-reſiſting medium, the difference 379 feet is the part loſt by the reſiſtance. In the equation of the time $t=A-n$, the arc n vaniſhes when $u=o$; and the radius $r^{\frac{1}{2}}=121.860$ is to the tangent $c=72.663$ as unity is to .596282, which anſwers to an angle of 30 degrees and 48 minutes, that is to 30.8 degrees. Now 180 degrees is to 30.8 as half the circumference $3.14159\sqrt{r}$ is to 65.507 feet for the arc A, and being divided by the ſquare root 5.67 of 32.2, by *Art.* 85, gives 11.55 ſeconds for the time t, which time in falling thro' the ſame height in a non-reſiſting medium, would have been 10.7 ſeconds, and the difference is .85 ſeconds.

REMARK.

It muſt be obſerved, that a falling body in a reſiſting medium can never acquire a velocity equal to that we have called the greateſt, *Art.* 158, but may perpetually approach it. For when $r=vv$, in *Art.* 166, both the ſpace deſcended thro' and the time of the

the fall become infinite; ſo that the greateſt velocity is only a limit to which the other velocities may approach without ever attaining it. This appears likewiſe, from obſerving, that both the acceleration and retardation or reſiſtance, are in proportion to the ſquares of the velocities; this being conſtantly the caſe, the diminution can never arrive to that degree as to become entirely equal to the augmentation: and conſequently the reſiſtance of falling bodies can never become geometrically equal to the force of gravity, nor the velocity acquired in the fall to the greateſt of all.

EXAMPLE.

168. If the body moves in a canal or horizontal right line with a given velocity; then becauſe $y=z$ in this caſe, and ſuppoſing gravity to be unity, the equation in *Art.* 163. becomes here $v\dot{v}=\dot{y}-\frac{vv}{r}\dot{y}$ or $\dot{y}=\frac{rv\dot{v}}{r-vv}$, whoſe fluent is $y=A-\frac{1}{2}r\log.\overline{r-vv}$; but when $y=o$, the velocity v becomes equal to the given one c; hence $A=\frac{1}{2}r\log.\overline{r-cc}$, and therefore the compleat fluent is $\frac{2y}{r}=\log.\frac{r-cc}{r-vv}$. Becauſe the equation of the time is $\dot{y}=v\dot{t}$, and as $\dot{y}=\frac{rv\dot{v}}{r-vv}$, by what has been ſhewn above, we get $\dot{t}=\frac{r\dot{v}}{r-vv}$; and if $r=aa$, the fluent of this equation will be $t=A+\frac{1}{2}a\log.\frac{a+v}{a-v}$; and when $t=o$, then $v=c$; hence $-A=\frac{1}{2}a\log.\frac{a+c}{a-c}$, and therefore $\frac{2t}{a}=\log.\frac{a+v}{a-v}-\log.\frac{a+c}{a-c}$ or $\frac{2t}{a}=\log.\frac{\overline{a-c}\times\overline{a-v}}{\overline{a+c}\times\overline{a-v}}$

COR:

C O R.

169. Hence it is manifeſt, that the given velocity, with which the body begins to move, muſt always be leſs than the greateſt velocity that the body can acquire in the medium. For when $r = ce$, or $a = c$, then both the ſpace moved over and the time elapſed vaniſh. Although this is very remarkable, yet no author has taken notice of it; on the contrary they often ſuppoſe this velocity much greater.

It muſt be obſerved that the ſame thing is true, whether the body moves in an horizontal line or in one that makes a given angle with it, ſince every thing will be the ſame as above. This we thought neceſſary to take notice of, to prevent an uncautious reader from taking what has been here ſaid as only applicable to horizontal ranges.

PROBLEM. *Fig.* 55.

170. *Suppoſing the powder all fired before the ſhot is ſenſibly moved from its place, to find the velocity with which it iſſues out of the piece.*

Let n expreſs the ratio of gravity to the force of powder, the height of the charge $AB = a$, any diſtance $AC = x$, $BD = b$; then becauſe the force of the powder confined in the ſpace AB is to its force, when extended to the ſpace AC, reciprocally as the ſquare of the diſtance AC to the ſquare of AB, by Cor. I. after Theo. IV. of our Artillery; that is, the force of the powder at C will be expreſſed by $\frac{aan}{xx}$. Now this value being wrote for p, and $\dot{x}$ for $\dot{y}$ and $\dot{z}$ into the general equation, *Art.* 163. as well as $\frac{vv}{r}$ for R, gives $v\dot{v} = \frac{aan\dot{x}}{xx} \times 1 - \frac{vv}{r}$ or $\frac{rv\dot{v}}{r - vv} = \frac{aan\dot{x}}{xx}$, whoſe fluent

ent is $A-\frac{1}{2}r \log. \overline{r-vv} = -\frac{aan}{x}$; but when x becomes $=a$, then v vaniſhes, ſo that $A=\frac{1}{2}r \log. r-an$. Hence $\frac{1}{2}r \log. \frac{r-o}{r-vv}=an-\frac{aan}{x}$, or becauſe $b=x-a$, we get $\frac{2abn}{rx}=\log. \frac{r-o}{r-vv}$. But if q denotes the number whoſe hyperbolic logarithm is $\frac{2abn}{rx}$, then $q=\frac{r}{r-vv}$, or $vv=\overline{q-1}\times\frac{r}{q}$.

C O R.

171. From whence it follows, in the ſame manner as in the laſt problem, that the velocity v muſt always be leſs and never can be equal to the greateſt velocity the ball may acquire in that medium: and conſequently the reſiſtance $\frac{vv}{r}$ of the body ariſing from its motion in the medium can never be equal, much leſs exceed the weight of the ſhot, as has erroneouſly been ſuppoſed by ſome authors.

OBSERVATIONS.

We have neglected the reſiſtance of the ſhot rubbing againſt the inſide of the gun, in the laſt problem, the force loſt thro' the vent and windage, and ſuppoſed the powder to fire all at the ſame inſtant, which is not true by experiments; beſides, if it was ſo, the ſhorteſt guns would be the beſt: but as we do not know by what law the flame proceeds, we are not able by theory alone to determine the exact velocity, the beſt length of a piece, or the proper charge. All therefore we are certain of is, that the velocity found by theory

is

is greater than it ſhould be, and yet leſs than that it may acquire in that medium: this appears from the equation $vv = \overline{q - 1} \times \frac{q}{r}$; for unleſs q is ſo great that unity may be neglected, we cannot have $vv = r$.

Again, if we conſider that the hyperbolic logarithm of $\frac{2abn}{rx}$, or $\frac{ab}{x}$, becauſe $2n$ and r are conſtant numbers, ſhould have certain limits to determine the length and charge: which it has not, for the rectangle made by two lines a, b when the ſum x of their length is given, is the greateſt of all, when they are equal, that is when the powder fills up half the vacant cylinder, which we know not to be true from undeniable experiments.

Tho' the theory is not ſufficiently exact, yet it ſerves as a guide to experiments; for which reaſon it ought to be well underſtood by thoſe who undertake to improve artillery.

Before this theory can be applied to particular examples, the ratio of the elaſtic force of powder to that of gravity muſt be determined. Let the preſſure of the atmoſphere be equal to a column of quickſilver of 29. 5 inches high: then as the ſpecific gravity of quickſilver is to that of caſt iron as 14000 to 7425, the preſſure of the atmoſphere will be equal to a column of iron 55.6 inches high, or 4.635 feet. Whence if we ſuppoſe with Mr. Belidor and moſt authors, Mr. Robins excepted, the elaſtic force of powder to be 4000 times greater than that of air; if we multiply 4.635 by 4000, we get 18540 feet for a column of iron, expreſſing the elaſtic force of powder: and as an iron ball, whoſe diameter is denoted by d, is equal to a cylinder of the ſame baſe and $\frac{2}{3}d$ high, the force of gravity is therefore to the force of powder as $\frac{2}{3}d$ is

to

to 18540, or as unity to $\frac{27810}{a}$, when the diameter d is expreſſed in parts of a foot.

EXAMPLE.

172. *Let an eighteen pounder of nine feet long be loaded with nine pound of powder, to find the velocity at the mouth of the cannon.*

The diameter d of an eighteen pound iron ball is five inches; and the height a of the charge ten inches or twice the diameter; then if we multiply 27810 by $\frac{5}{12}$, we get $n=66744$, $x=9$, $b=8\frac{1}{6}$; and as the ſpecific gravity of caſt iron is to that of air as 7425 is to $\frac{7}{8}$, we get $r=21214$, *Art.* 159; hence $\frac{2abn}{rx}=4.758$, for the hyperbolic logarithm of the number q; and this number multiplied by the conſtant number .43429, *Art.* 59. gives 2.06635 for the tabular logarithm of that the number, which gives $q=116.5$.

Whence $vv=q-1\times\frac{r}{q}=21032$, or 823 feet per ſecond for the velocity required; which differs from the greateſt 826 by 3 feet only. Tho' this agrees very exactly with the experiments made at Minorca, as will be ſeen hereafter, when we treat of projectiles in the air, yet it does not anſwer in other examples; for the reaſons given in the obſervations above.

REMARK I.

What has hitherto been ſaid in this work, was only to prepare the reader to underſtand what relates to Artillery and Gunnery, as being the ſubject for which it was chiefly intended, and could not be explained without the preceding principles.

Notwithſtanding the great importance of artillery to a ſtate, neither the beſt length nor charge of guns, ſo as to carry their ſhot to the greateſt diſtance, have as yet been determined to any tolerable degree of perfection. This is not to be wondered at, becauſe all the experiments made here and abroad were ſo injudiciouſly contrived, that no true concluſions could be made from them; to this may be added, that the practical artilleriſts are entirely ignorant of the theory, and the theoriſts of the practice; and without being acquainted with both, it is obvious that no improvements can be made in this ſcience.

Authors, for want of having a proper guide to go by, have greatly over rated the velocities of cannon ſhot, as likewiſe the reſiſtance they meet with moving in the air; and as long as this error ſubſiſted, no improvement could be expected. For we have proved in *Art.* 171, that a body impelled by any force whatever can have no greater velocity than that it may acquire by falling in that medium, which we have ſhewn, in *Art.* 159, cannot exceed a certain limit, which we have determined by means of the ratio between the ſpecific gravities of the medium to that of the body. Thus if the body is of caſt iron, its ſpecific gravity is to that of air as 7425 to $\frac{7}{8}$, by the Remark after *Art.* 147; whence if we make $m=7425$, and $n=\frac{7}{8}$ in the equation, $r=\frac{6md}{n}$, *Art.* 159. we get $r=50914d$; which ſhews that twice the height thro' which an iron ball deſcends ſo as to acquire the greateſt velocity, is leſs than 50914 times its diameter expreſſed in feet; it ſhews likewiſe that the greateſt velocities which iron balls can acquire are in proportion to their diameter.

From

From whence we can correct an error commonly committed in the conſtruction of guns; which is, that thoſe of ſmaller calibers, under a twenty-four pounder, are made longer in proportion than larger ones. The practitioners imagine, that by this means the velocities of their ſhot may be increaſed, and go as far as larger ones, by loading them with greater charges. A few examples will prove what we here advance.

Let the cannon be a ſix pounder, whoſe diameter of the ſhot is 3.498 inches, then 50914, multiplied by 3.498, and divided by 12, gives 14841.4 feet; this multiplied by 32.2, and the ſquare root of the product extracted, gives 691 feet for the ſpace moved over uniformly by the greateſt velocity a ſix pounder ſhot can poſſibly have in a ſecond.

If the cannon be a forty two pounder, whoſe diameter of the ſhot is 6.684 inches, then we get $r = 28359$ feet; this multiplied by 32. 2 and the ſquare root extracted from the product, gives 955 feet in a ſecond for the greateſt velocity a forty-two pound ball can poſſibly have.

When the ball is of lead, then as the ſpecific gravity of lead is to that of air as 11325 to $\frac{7}{8}$; the equation $r = \frac{6md}{n}$ gives $r = 77657d$ feet, for the ſhot whoſe diameter d is expreſſed in feet. Thus for example, the diameter of a bullet, twelve of which weigh a pound, is $\frac{3}{4}$ of an inch nearly; ſo that writing $\frac{1}{10}$ for d, we get $r = 4854$ feet nearly; this multiplied by 32.2, and the root extracted from the product, gives 395 feet in a ſecond, for the greateſt velocity which that ſhot can have.

Again, let the diameter of the ſhot be 1.69 inches, whoſe weight is a pound; then $r = 77657d$ becomes

$r = 10937$ feet in this caſe, and proceeding in this manner as before we get 593 feet in a ſecond, for the greateſt velocity this ſhot can have.

As a cannon ſhot weighing 55 pounds, whoſe diameter is 7.32 inches nearly, will by proceeding as before, give 1000 feet in a ſecond for the greateſt velocity this ſhot can have, and as no ſhot of this ſize has hitherto been uſed, nor will hardly hereafter. we may draw this concluſion, that no cannon or muſket ſhot can acquire a velocity, by any force impreſſed upon them, of a 1000 feet in a ſecond; conſequently thoſe authors who have ſuppoſed, that they may have a velocity of 2000 feet and upwards, in a ſecond, were greatly miſtaken: and whatever has been deduced from this ſuppoſition muſt be erroneous.

It will not be improper to take notice here of a gun which the French made uſe of in the laſt war in Germany called *Amuſette*, which fired a leaden ball of a pound weight; and it was ſaid that it carried its ſhot farther than any of our light ſix pounders. Now, we have ſhewn, that the greateſt velocity of a ſix pound ſhot approaches to 691 feet in a ſecond, and that the greateſt velocity of a leaden ball weighing one pound, approaches to 593 feet in a ſecond; conſequently, if the fact be true, our ſix pounders have not their proper length, nor charged with a proper quantity of powder. This appears likewiſe from other circumſtances, as will be ſhewn hereafter.

We have hitherto ſpoke of ſolid ſhot, we ſhall now conſider the velocities of ſhells; whoſe ſpecific gravity is to that of air as 4892 to $\frac{7}{8}$, ſuppoſing them all ſimilar to the thirteen inch ones, as we have done in our general conſtruction: hence $r = \frac{6md}{n}$, gives

gives $r=33545d$, in reſpect to all ſhells made of caſt iron. Now, as the diameter of a thirteen inch mortar ſhell is $12\frac{3}{4}$ inches, we get $r=35641$ feet, and proceeding in the ſame manner as before, we ſhall find 1072 feet in a ſecond, for the greateſt velocity which this ſhell may have.

The diameter of a ten inch mortar ſhell being $9\frac{3}{4}$ inches, we get $r=27255$ feet, and 936.8 feet in a ſecond, for the greateſt velocity. Hence the ſame thing is true in regard to ſhells as in ſolid ſhots; that is, the greateſt ſhells may have the greateſt velocities.

REMARK II.

For the reaſon that authors have ſuppoſed the velocities of bodies moving in a reſiſting medium, may increaſe without any limits; the reſiſtance may likewiſe increaſe without end, which is a great miſtake, as we are going to prove. Suppoſe the denſity of the air is the ſame every where as near the ſurface of the earth, and a body was to fall in it, by the force of gravity; it is proved, that if there was no reſiſtance, it would deſcend through ſpaces which are as the ſquares of the velocities acquired in the fall; but, by reaſon of the reſiſtance of the air, the ſpaces deſcribed will be diminiſhed in ſome proportion, which can be no other than that of the ſquares of the velocities: ſince the number of particles of the air which the body ſtrikes in a given time, are as the velocities, and as the body ſtrikes each particle with a force that is likewiſe as the velocity, the reſiſtance ariſing from theſe cauſes muſt therefore be as the ſquares of the velocities. In oily or unctuous mediums, the reſiſtance will be increaſed in proportion to the difficulties of ſeparating the particles; which is not the caſe in the air. Now the deſcent of bodies

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in the air, being accelerated in the duplicate ratio of the velocities, and reſiſted in the ſame duplicate ratio; the body can never ariſe to ſuch a degree of ſwiftneſs, as that the force of reſiſtance becomes equal to the force of gravity, altho' it may continually approach it. If theſe two forces ſhould become equal at laſt, the motion would then become uniform; this has been ſhewn in *Art.* 153. and 158. conſequently, the force of reſiſtance can never become equal, much leſs greater, than the force of gravity in this caſe.

What has been ſaid here in reſpect to falling bodies, is equally true in bodies impelled by any force in any direction, by *Art.* 169. and 171; beſides, the force by which the body is impelled, let it be what it will, cannot be inſtantaneous, and muſt act gradually by degrees; and whenever its action becomes equal to that of the reſiſtance, it cannot increaſe beyond it. Conſequently, the notion, that the reſiſtance may increaſe without any limit, is without foundation; and thoſe experiments that have been made on that account muſt be fallacious, and can avail nothing,

An objection is made againſt the reſiſtance being in all caſes as the ſquares of the velocities; when the motion of the body is ſo great, as that the air cannot fill up immediately the ſpace left by the ſhot, the reſiſtance muſt be greater than it would be in ſlower motions; from thence is inferred, that the reſiſtance is, in ſlow motion, as the ſquare of the velocity, but in very ſwift motion, in the triplicate ratio of the velocities.

But as in the eſtimation of the reſiſtance, the force by which the motion of bodies are retarded, was determined from the number of the particles the body ſtrikes, and from the velocity with which they are ſtruck, and not from any conſideration of their moving round

round and ſtriking the body behind; ſo that where this laſt action takes place, and is ſufficient to produce an effect that can be eſtimated, the contrary concluſion to that objection may be made, viz. that in ſlow motion the reſiſtance is diminiſhed, by reaſon of the particles ſtriking the body behind in the direction of its motion. This ſeems to be confirmed by experience, for in ſlow motion the curve deſcribed by the body is ſenſibly a common parabola. This will be found by meaſuring the ranges, as I have done ſeveral times at La Fere; and always found in diſtances not exceeding 300 yards, that the ranges never differed above a few feet from that computed by means of the parabola.

The next query is, whether a ſhot can move ſo faſt as to leave any vacuity behind it, or not? If not, the objection is without foundation. Now, to find this recourſe muſt be had to experiments. We have ſhewn in *Art.* 147, that the ſound moves over a ſpace of 1142 feet in a ſecond; and as the ſound is propagated by the elaſtic force of the air, therefore the elaſtic force of the air is ſuch as to produce a velocity which moves over the ſame ſpace in the ſame time: we have likewiſe proved in the laſt remark, that a thirteen inch ſhell may have the greateſt velocity of any ſhot or ſhell that are now in uſe; and as this velocity is ſuch as to move over a ſpace of 1072 feet in a ſecond, when uniformly continued, which being leſs than the velocity produced by the elaſtic force of air, it is manifeſt, that no common ſhot can move ſo quick as to leave a vacant ſpace behind it.

REMARK III.

Having determined the greateſt velocities that cannon ſhot can poſſibly have, let the charge be what it will, in the firſt remark; and the greateſt reſiſtance they

they can meet with in their motion, in the ſecond; it remains now to apply theſe principles to the improvement of artillery, which were neceſſary to be known; and for want of them authors have been greatly miſled in their computations and conjectures.

Of all queſtions that ever have been made in reſpect to artillery, the moſt important ones are thoſe concerning the proper length of guns and their beſt charges, ſo as to carry their ſhot as far as poſſible; but authors, as well as practitioners, differ widely in their opinions: which could be no otherwiſe; for the practitioners, for want of theory, have been miſguided in their inquiries, and moſt experiments that were made are ſuch as no concluſions that are really uſeful can be drawn from them; for thoſe made at Dunkirk by Dumetz, with various calibers of the ſame length, and with charges weighing two-thirds of their ſhot, can anſwer no other purpoſe than that a twenty-four pounder carries its ſhot farther than any other caliber of the ſame length; and thoſe made at Woolwich by the late General Armſtrong, with ſix twenty-four pounders of various lengths, can ſerve only to ſhew that the piece which was 9.5 feet carried farther than thoſe of any other length with the ſame charge. Theſe are the moſt remarkable experiments that I know off, excepting thoſe made at Minorca by General Williamſon, and are the only ones to the purpoſe, as will be ſhewn herafter.

In the firſt period of time, guns were made very heavy and long, ſo as to bear charges equal in weight to that of their ſhot, imagining that the greater the charge was, the longer the gun ſhould be, in order to give time for the powder to take fire before the ſhot leaves the piece, and to produce the greater effect. In Queen Elizabeth's time, we and the French made them of exceſſive length, as appears by the cul-

verin in Dover Caſtle, and that at *Nancy*; whilſt the Spaniards made them ſhorter and lighter than they have been ever ſince. For at the ſiege of Breda by the Spaniards, under the command of Spinola, in 1625, Count Mansfield, a German in their ſervice, caſt ſeveral ſix pounders which weighed 180 lb. only, and twenty-five pounders of 700 lb. weight, beſides another Spaniſh gun, which we had at Woolwich, that weighed 2100, and carried a ball of 41 lb. of our weight. In King Charles the Second's time, we made thirty-two pounders of iron that weighed 4200, and forty-two pounders of 5300 weight; whereas our preſent twenty-four pounders braſs weigh 5200, and the forty-two of the ſame metal, 6600. To make braſs guns ſo much heavier than the iron ones of the ſame caliber, is inexcuſable in the land of liberty, where reaſon and good ſenſe prevails in almoſt every thing elſe.

It is true we make at preſent our ſix pounders for the field-ſervice ſo light and ſhort, that four of them weigh no more than one of the long and uſual guns; but we are to thank the late Duke of Cumberland for it, who, by his authority and judgment in artillery, inſiſted upon the diminiſhing the weights. This great variation in the length of guns ſufficiently ſhews how little improvement has been made in artillery ſince its firſt invention.

The theoriſt having no proper experiments, and being unacquainted with the practice, it is not ſtrange that he ſhould be ſo much miſtaken in his calculations; but that he ſhould fall into the vulgar error of the practitioners, by admitting the velocities to increaſe in proportion as they are charged with a greater quantity of powder, without any limitation, is inexcuſable. As they all admit the reſiſtance which moving bodies meet with, one ſhould think it required no great degree of penetration to

perceive that this reſiſtance muſt diminiſh the velocities in ſome caſes, ſo as not to be augmented.

What can be the reaſon that no nation has endeavoured to improve their artillery, which is eſſentially uſeful and ſo very expenſive, unleſs it is the prepoſſeſſion in favour of an old eſtabliſhed cuſtom, in ſuch a manner, as neither reaſon or good ſenſe can overcome? It will therefore be in vain for an author to endeavour all he can for improving artillery, unleſs ſome great man, who has it in his power to undertake it, and to give proper encouragement to thoſe that are qualified for carrying on a ſeries of proper experiments.

To facilitate ſuch an undertaking, we ſhall here inſert the experiments made at Minorca, in 1745, with two iron eighteen pounders, one of eleven feet long, weighing 51 : 0 : 5, and the other nine feet long, and weighing 39 : 1 : 3, under the direction of General Williamſon, aſſiſted by Major Hiſlop and ſeveral other artillery officers.

Theſe two pieces were firmly fixed in the ground, and ſo contrived as to have any degree of elevation; they were loaded with different charges and elevations, from ſix to forty-five degrees. After having made a great many trials, it was found, that, when they were loaded with nine pounds of powder, they both carried their ſhot fartheſt, in all the different elevations; and that the gun of nine feet long carried always farther than that of eleven feet long. Theſe experiments being authentick, furniſh a ſure direction to determine the proper length of guns, and their beſt charges. For as the diameter of an eighteen pound ſhot is 5 inches nearly, this gun is $21\frac{3}{8}$ diameters of the ſhot long; and therefore the beſt length of any gun is about 21 diameters of its ſhot, and its beſt charge equal in weight to half that of its ſhot. But before a ſtate is reſolved to reform

their

their artillery, it would be neceſſary to repeat thoſe experiments, and try various calibers in the ſame manner, in order to be certain that this alteration, in ſo eſſential a thing, may be done upon ſolid and firm conſiderations; and when the dimenſions of guns are fixed, a law ſhould be made to prevent any one to alter them for the future.

Having thus aſcertained the limits of the greateſt velocities, as well as the reſiſtance of bodies, the ranges of ſhells and ſhots may be determined in a greater degree of exactneſs than by any other theory yet publiſhed; and as their computation is very intricate and laborious, tables may be conſtructed ſo as to apply the theory to practice with eaſe.

PROBLEM. *Fig.* 56.

173. *If a body projected with a given velocity c, in a given direction* AE, *to find the nature of the curve* ADK *deſcribed by the body.*

If AP$=x$, PM$=y$, the arc AM$=z$, v the velocity at M, t the time and $p=1$; then $\dot{v}v=-\dot{y}-R\dot{z}$, by *Art.* 163. for the aſcent, and $\dot{x}\dot{y}=v\dot{v}\ddot{x}$, by *Art.* 164. when $\dot{z}$ is conſtant, and as $R=\frac{vv}{r}$, by *Art.* 165. If the values of $\dot{y}$ and R are wrote into the equation above, we get $\dot{v}v=-\frac{vv\ddot{x}}{\dot{x}}-\frac{vv\dot{z}}{r}$, or dividing by vv, and tranſpoſing the firſt term of the ſecond ſide to the firſt, we ſhall have $\frac{\dot{v}}{v}+\frac{\ddot{x}}{\dot{x}}=-\frac{\dot{z}}{r}$, whoſe fluent is log. $\frac{v\dot{x}}{\dot{z}}=-\frac{Az}{r}$; becauſe $\dot{z}$ has been ſuppoſed conſtant, it was neceſſary to divide by it, to make the terms homogeneous: and if q expreſſes the number whoſe hyperbolic logarithm is unity, that is, $lq=1$;

then $\log.\frac{v\dot{x}}{\dot{z}} = -\frac{Az\lambda q}{r}$, and hence by logarithms $\frac{v\dot{x}}{\dot{z}} = Aq^{-\frac{z}{r}}$. Now at the beginning A of the motion $v = c$, $z = o$; and if the radius is to the cosine of the angle EAP as unity to k; then $\dot{z} : \dot{x} :: 1 : k$, we get $A = ck$, consequently the last equation becomes $\frac{v\dot{x}}{\dot{z}} = ckq^{-\frac{z}{r}}$.

And since $\dot{z}^2 + vv\ddot{y} = o$, by *Art.* 164, when $\dot{x}$ is constant, and if for convenience we make $\dot{x} = 1$; by comparing the values of vv, in the two last equations, we get $q\frac{2z}{r} + cckk\ddot{y} = o$, for the equation of the curve.

174. Since $\dot{z} = \nu\dot{t}$, by *Art.* 161, and $v\dot{x} = ck\dot{z}q^{-\frac{z}{r}}$, by above; the two values of v being compared, give $\dot{x}q^{-\frac{z}{r}} = ck\dot{t}$. Hence if we suppose the resistance to vanish, r becomes infinite, and therefore $q^{-\frac{z}{r}} = 1$; so that the equation $\frac{v\dot{x}}{\dot{z}} = ckq^{-\frac{z}{r}}$, becomes $v\dot{x} = ck\dot{z}$, and $\dot{x}q^{-\frac{z}{r}} = ck\dot{t}$, gives $\dot{x} = ck\dot{t}$, or $x = ckt$, which values of v and t are exactly the same as we have found in Articles 94,95, where we have treated of bodies moving in a non-resisting medium.

Several authors have found the equation of the curve, nearly expressed like this here; but in finding the relation between the abscissa and ordinate, no one that I have as yet seen has done it in a general manner. Thomas Simpson has, it is true, an equation that will answer when the projectile velocity is but small; but when it is great, he has recourse to the differential method. But as some quantities are to be found by trials, which in many cases is not to be done without an infinite deal of labour, it is obvious that his method is often impracticable.

Professor

Profeſſor Euler ſolved this equation by an infinite ſeries in his Mechanics, as likewiſe in his Comment upon Robins's Gunnery, but neither of them converge ſo as to anſwer all caſes; we ſhall here give three different ſolutions of this problem, and it will appear that they all give very nearly the ſame ranges.

FIRST METHOD.

PROBLEM.

175. *To ſolve the equation* $q^{\frac{2z}{r}} + cckk\ddot{y} = o$, *by an infinite ſeries.*

The fluxion of $-q^{\frac{2z}{r}} = cckk\ddot{y}$ is $-\frac{2\dot{z}}{r}q^{\frac{2z}{r}} = cckk\dddot{y}$, and theſe two equations multiplied croſſways, ſo as to expel the exponential quantity, gives $r\dddot{y} = 2\dot{z}\ddot{y}$.

If now we ſuppoſe y equal to an infinite ſeries compoſed of the powers of x and conſtant but indetermined coefficients a, b, c, d, f, and make $\dot{x} = 1$, for convenience's ſake; then the firſt, ſecond, and third fluxions, give

$$y = ax + bx^2 + cx^3 + dx^4 + fx^5$$
$$\dot{y} = a + 2bx + 3cx^2 + 4dx^3 + 5fx^4$$
$$\ddot{y} = 2b + 6cx + 12dx^2 + 2afx^3$$
$$\dddot{y} = 6c + 24dx + 60fx^2.$$

As the ſquare of $\dot{y}$ added to the ſquare of $\dot{x}$, is equal to the ſquare of $\dot{z}$; by making $1 + aa = ss$, we get $\dot{z}^2 = ss + 4abx + \overline{6ac + 4bb} \times x^2$, and if $m = \frac{3ac}{s} + \frac{2bb}{s^3}$; the ſquare root of the laſt equation will be $\dot{z} = s + \frac{2abx}{s} + mx^2$. Hence, by writing the values

of

of $\dot{z}$, $\ddot{y}$ and $\dot{\ddot{y}}$, into the equation $r\dot{\ddot{y}} = 2\dot{z}\ddot{y}$, we get,

$$3cr + 12rdx + 30rfxx = \left.\begin{array}{l} 2bs + 6cs \\ +\frac{4abb}{s} \end{array}\right\} x \quad \left.\begin{array}{l} +12ds \\ +2bm \\ +\frac{12abc}{s} \end{array}\right\} xx.$$

And by comparing the coefficients of the homologous terms, then will

$$3rc = 2bs, \qquad \text{or } 3c = \frac{2bs}{r}$$

$$6rd = 3cs + \frac{2abb}{s} \qquad 3d = \frac{bss}{rr} + \frac{abb}{rs}$$

$$15rf = 6ds + bm + \frac{6abc}{s} \qquad 15f = \frac{2bs^3}{r^3} + \frac{8abb}{rr} + \frac{2b^3}{rs^5}.$$

Conſequently the value of y will be expreſſed by the powers of x and known quantities, when the coefficients a and b are determined: in order to which we are to conſider, that at the beginning A of the motion, where $x = o$, the ſecond equation gives $\dot{y} = a$, or $\dot{y} = a\dot{x}$, becauſe $\dot{x}$ has been ſuppoſed unity. Hence, $\dot{x} : \dot{y} :: 1 : a$; but $\dot{x}$ is to $\dot{y}$ as the radius to the tangent of the angle EAP of elevation; therefore a denotes the tangent of that angle, when the radius is unity, and becauſe $1 + aa = ss$, s expreſſes its ſecant.

The third equation gives $\ddot{y} = 2b$, when $x = o$, and writing $2b$ for its equal $\ddot{y}$ into the equation $\dot{z}^2 + vv\ddot{y} = o$, *Art.* 164, we get $\dot{z}^2 + 2bvv = o$, but at A, v becomes c and $\dot{z}$ the ſecant s; ſo that $ss + 2bcc = o$, or if h expreſſes double the height thro' which a falling body acquires the projectile velocity c, then $h = cc$, and $ss + 2bh = o$.

As the values of y converges but ſlowly in many caſes, without ſome alteration being made in the equation, the moſt ſimple and ſhorteſt way to do this, is to write hx and hy, for x and y; or which is the ſame, by multiplying the coefficients of each term,

term, by the powers of h one leſs than that of x; that is, a, b, c, d, f, by 1, hh, h^3, h^4, and making $\frac{h}{r}=n$, which gives $y=ax-bx^2-cx^3-dx^4-fx^5$, and $2b=ss$, $3c=ns^3$, $12d=2nns^4-ans^3$, and $60f=4n^3s^5-8an^2s^4+ns^3$.

It muſt be remembered, that when the values of x and y are found by this equation, to multiply them by the value of h, according to the ſuppoſition we have made, to get their true values. Hence, when the ratio n of the greateſt and projectile velocities is given or remains the ſame, the values found for x and y will ſerve to determine the various parts of the projectiles in general, and if after the diameters of the ſhots are given, the particular caſes of each will be obtained; which renders the equation very general; and it will be ſhewn hereafter that the five firſt terms are ſufficient to find the value of x very nearly.

C O R.

176. When the reſiſtance becomes inſenſible, all the terms divided by r, or, which is the ſame, multiplied by n, vaniſh, and therefore the equation becomes in this caſe $y=ax-bx^2$, which is preciſely the ſame as that found in *Art.* 98, becauſe $ak=l$, $2bh=ss$, and h is here double to what it is there.

C O R.

177. When the body is projected in an horizontal direction, the tangent a vaniſhes, the ſecant s becomes equal to the radius or unity, and y negative. Therefore $y=bx^2+cx^3+dx^4+fx^5+gx^6+kx^7$; and $2b=1, 3c=n, 6d=nn, 60f=4n^3+n, 180g=4n^4+7nn, 315k=2n^5+16n^3-\frac{3}{8}n$.

The equations $vq\frac{z}{r}=ck\dot{z}$, $q\frac{2z}{r}+cckk\ddot{y}=o$, found in *Art.* 173 and 164, become here $vq\frac{z}{r}=c\dot{z}$ and $q\frac{2z}{r}=h\ddot{y}$; becauſe the ſign k is here unity, $h=cc$, and $\ddot{y}$ negative;

negative; but it muſt be obſerved that c denotes the projectile velocity at the vertex D where the motion is ſuppoſed to begin.

C O R.

178. Since $\dot{x}=vt$ and $\dot{x}^2+vv\ddot{y}=o$, by *Art.* 164, we get $-\ddot{y}=\dot{t}$. Now as the third equation $\ddot{y}=2b+6cx+12dx^2+20fx^3$, gives $-\ddot{y}=ss+2ns^3x+12dx^2+20fx^3$, becauſe $-2b=ss$, and $-3c=ns^3$. And if $2ms=12d-nns^4$; $25p=20f-2mnss$; the ſquare root of this equation will be $\dot{t}=s+nssx+mx^2+px^3$; whoſe fluent is $t=sx+\frac{1}{2}nssx^2+\frac{1}{3}mx^3+\frac{1}{4}px^4$.

The velocity v is found by writing the ratio of $\dot{x}^2$ to $-\ddot{y}$ into the equation $\dot{x}^2+vv\ddot{y}=o$.

PROBLEM. *Fig.* 56.

179. *The angle of elevation being* 45 *degrees, and the ratio n unity, to find the horizontal range* AK.

Here we have $a=1$, $ss=2$, and $n=1$, by ſuppoſition; and hence $b=1$, $c=.9428$, $d=.4309\frac{2}{3}$, $-f=.10907\frac{1}{3}$. Therefore the equation in *Art.* 175, becomes here $y=x-x^2-.9428x^3-.431x^4+.109x^5$. Now when $y=o$, then $1=x+.9428x^2+.431x^3-.109x^4$. And by the general rule for finding the roots, *Art.* 78, we get 1 : 1 : 1.9428 : 3.3166 : 5.4703 : 9.3255 : 15.7 : 26.4883 : 44.7133 : 75.4367 : 127.2975 : 214.8034, and the penultiem term divided by the laſt gives $x=.59262$, true in all its places. For $x^2=.351198$, $x^3=.2081269$, $x^4=.12334$, and theſe values wrote into the equation gives $1=1.0000$.

PROBLEM.

180. *To find the abſciſſa* AB, *correſponding to the greateſt ordinate* BD.

By taking the fluxion of the laſt equation, and making $\dot{y}=o$, we get $1=2x+2.8284x^2+1.7238x^3-.5454x^4$. By finding the root as before, then 1 : 2 : 6.8284 : 21.0374 : 64.2904 : 198.763 : 611.905 : 1885.3413; and $x=.32456$, nearly. For $x^2=$ .10534,

10534, $x^3 = .034189$, $x^4 = .011096$; and theſe values wrote into the equation gives $1 = .999946$.

PROBLEM.

181. *To find the greateſt ordinate* B D.

If we write the values of x and its powers here found into the equation $y = x - x^2 - .9428\,x^3 - .431\,x^4 + .109\,x^5$, we get $y = .1826$.

EXAMPLE.

182. We have found $r = 50914\,d$, in the firſt remark after *Art.* 172; when the ſhot is of caſt iron: now, if it weighs 18 pounds, the diameter d will be five inches, and hence $r = 21214$ feet, and as b has been ſuppoſed equal to r; if we multiply 21214 by .59262, the value of x found in art. 179, we get 12572 feet or 4190 yards for the greateſt horizontal range A K, which an 18 pounder can have when elevated at an angle of 45 degrees.

If the value of x (.32456) in *Art.* 180, be multiplied by 21214, we get 6885 feet or 2295 yards for the abſciſſa A B correſponding to the greateſt ordinate, and .1826, multiplied by 21214, gives 3873.6 feet or 1291.2 yards for the greateſt ordinate B D.

REMARK.

In order to ſhew the near agreement of our Theory with the beſt experiments made by General Williamſon, &c. at Minorca in 1745, as mentioned before; who, after many trials with various charges, found that nine pounds of powder carried an 18 pound ball farther than any other charge: having alſo tried two 18 iron pounders, one of nine and the other eleven feet long, found that of nine feet long carried its ſhot farther than that of eleven, under the ſame circumſtances: after this, he ſet down ſix

experiments, made with that of 9 feet long, loaded with 9 pounds of powder, and elevated at an angle of 45 degrees: their ranges were found as in the margin.

4160 yards,
4135
4100
4015
3980

This ſhews how nearly the range 4190, we found in *Art.* 182, by our theory agrees with theſe experiments; for we have ſuppoſed a greater projectile force than the ſhot could have, as ſhewn before: and hence it is plain, that the ſpecific gravity of air is $\frac{7}{8}$ nearly, as we determined it after, *Art.* 147. For if we ſuppoſe it to be $\frac{5}{4}$, as Cotes has it, then $r = \frac{6\,m\,d}{n} = 14850$ feet, which multiplied by .59262, the value of x, gives 2933 yards; this being leſs than any of thoſe above, ſhews that this ſuppoſition is contrary to theſe experiments, ſince this range ſhould be rather more than leſs, becauſe we have ſuppoſed a greater projectile velocity than the ſhot could have.

If we make uſe of Sir Iſaac Newton's rule $r = \frac{8\,m\,d}{3\,n}$ for determining the greateſt velocity, and ſuppoſe $n = \frac{7}{8}$ according to our rule, we ſhall find $r = 9429$ feet, which when multiplied by .59262, the value of x, we get 1862.6 yards for the horizontal range, which plainly proves this rule to be deficient likewiſe: and if we were to make $n = \frac{5}{4}$, as Cotes has it, the range comes out to be 1303.4 yards only; and conſequently our determination of the ſpecific gravity of the air, and that of the reſiſtance, is the neareſt that poſſibly can be found.

In all arts and ſciences of a mixt nature, a criterion is abſolutely neceſſary to try their certainty; for the leaſt error, either in the facts or ſuppoſitions, miſleads an author in ſuch a manner as that the concluſion

cluſion comes out quite contrary to what it ought to be: this has been the caſe of all the authors that have wrote upon motion in a reſiſting medium, for want of having good experiments to compare with their calculations: as to thoſe experiments which ſome of them made, they were too diminutive to draw proper conſequences from them, whereby they could not eſtabliſh their theory upon a proper foundation.

PROBLEM.

183. *To find the velocity at the higheſt point* D, *and the time of deſcribing the arc* A D.

We have $\ddot{y} = 2b + 6cx + 12dx^2 + 20fx^3$, by *Art.* 175; and as $-b = 1$, $-c = .9428$, $-d = .4309\frac{2}{3}$, $+f = .10907\frac{1}{3}$; the equation becomes $-\ddot{y} = 2 + 5.6568x + 5.1716x^2 - 2.1816x^3$: and if we write the values of x and its powers found in *Art.* 180, we get $-\ddot{y} = 4.30616$; and as $\dot{z}$ is here unity, the equation $\dot{z}^2 + vv\ddot{y} = o$, gives $vv = .2322$, which multiplied by 21214, the value of h, gives 4925.89 feet for twice the height through which a falling body acquires the velocity ſought: and if we multiply this height by 32.2, twice that of a body fallen through in a ſecond, and extract the ſquare root of the product, we ſhall have 398 feet moved over uniformly in a ſecond by this velocity.

By a method which will be given hereafter, for finding the velocities independant of the foregoing infinite ſeries, we find $vv = 2328$; which differs therefore by .0006 parts from that above: this ſhews how nearly the values of x have been found by the former operations.

We have $t = sx + \frac{1}{2}nssx^2 + \frac{1}{3}mx^3 + \frac{1}{4}px^4$ by *Art.* 178, $2ms = 12d - nns^4$, $2ps = 20f - 2mnss$; and becauſe $ss = 2$, $n = 1$, $d = .4309\frac{2}{3}$, $-f = .10907\frac{1}{3}$, we get $m = .4142$, $-p = 1.357$: therefore $t = 1.4142x + x^2 + .1381x^3 - .339x^4$. Now if we

write the values of x and its power found in *Art.* 180, we get $t = .565$. But if this value be multiplied by the ſquare root of h (21214) we ſhall have the height fallen through in that time, and if that height be divided by the ſquare root of 32.2, twice the height fallen through in a ſecond, we ſhall have the number of ſeconds imployed in the fall, by $2s = ptt$ in *Art.* 84.

If therefore we divide 21214 by 32.2, and extract the ſquare root of the quotient, we get 25.667, which, multiplied by .565, gives 14.5 ſeconds for the time required.

PROBLEM.

184. *To find the velocity with which the ſhot ſtrikes the ground at* K, *and the time of deſcribing the whole curve.*

We have $-\ddot{y} = 2 + 5.6568x + 5.1716x^2 - 2.1816x^3$, by *Art.* 183; and by writing the values of x and its powers found in *Art.* 179, we get $-\ddot{y} = 6.7145$. But the value of z^2 muſt be known before we can find that of vv: as we have $\dot{y} = 1 - 2x - 2.8284x^2 - 1.7238x^3 + .5454x^4$, by *Art.* 180; by writing the values of x and its powers found in *Art.* 179, we get $-\dot{y} = 1.47026$, whoſe ſquare added to unity gives $z^2 = 3.1609$; and by $\dot{z}^2 + vv\dot{y} = o$, we get $vv = .4558$, and this multiplied by 21214, the value of h, gives 9669 feet for twice the height through which a falling body acquires this velocity; or if this number be multiplied by 32.2, the ſquare root of the product, gives 558 feet moved over uniformly in a ſecond by this velocity.

We have $t = 1.4142x + x^2 + .1381x^3 - .339x^4$; and by writing the values of x and its powers in *Art.* 179, gives $t = 1.176$; and this multiplied by 25.667, *Art.* 183, gives 30 ſeconds for the time required.

It

It is evident that the value 1.47026 of $\dot{y}$ is the tangent of the angle in which the ſhot ſtrikes the ground, which therefore is 55 degrees and 47 minutes.

PROBLEM.

185. *Let the projectile velocity be half the greateſt poſſible, and the reſt the ſame as before, to find the horizontal range* A K.

As $n = \frac{1}{2}$ by ſuppoſition, we have $c = .4714$, $d = .0488$, $-f = .0626$; and therefore the general equation becomes $y = x - xx - .4714x^3 - .0488x^4 + .0626x^5$, and when $y = o$, then $1 = x + .4714x^2 + .0488x^3 - .0626x^4$. Hence by the general rule for finding the roots 1 : 1 : 1.4714 : 1.9916 : 2.6714 : 3.6194 : 4.8845 : 6.5966 : 8.9087 : 12.03, and $x = .7405$ nearly. For if the value of x and its powers are wrote into the equation, we get $1 = 1.00000$.

PROBLEM.

186. *To find the abſciſſa* A B, *correſponding to the greateſt ordinate* B D.

The fluxion of the laſt equation, when $\dot{y} = o$, gives $1 = 2x + 1.4142x^2 + .1952x^3 - .3131x^4$, and finding the root of this equation, as uſual, we get $x = .4$ nearly.

In ſearching the velocity at the higheſt point D, I found $vv = .3179$, and by the method given hereafter $vv = .3177$; ſo that the difference is .0002 only; which ſhews how nearly the value of x has been found by the infinite ſeries.

PROBLEM.

187. *Let the projectile velocity be but one tenth of the greateſt, and the reſt as before, to find the horizontal range* A K.

Becauſe

Becauſe $n = .1$, by ſuppoſition, the general equation becomes here $y = x - x^2 - .09428x^3 + .0169x^4$; and when $y = o$; then $1 = x + .09428x^2 - .0169x^3$; hence we get 1 : 1 : 1.09428 : 1.17166 : 1.25793 : 1.3499 : 1.4487 : 1.5562 : 1.6699; and $x = .9318$ nearly.

PROBLEM.

188. *To find the abſciſſa* A B, *correſponding to the greateſt ordinate* B D.

By making $\dot{y} = o$, in the former equation, we get $1 = 2x + .2828x^2 - .0676x^3$, and 1 : 2 : 4.2828 : 9.0636 : 19.2032 : 40.68; hence $x = .472$ nearly. For $x^2 = .22278$, $x^3 = .10515$; and theſe values wrote into the equation give $1 = 1.000$.

COR.

189. Hence, if theſe values of x and its powers are wrote into the equation found in *Art.* 187, we get $y = .2405$ nearly for the greateſt ordinate.

And theſe values wrote into $-\ddot{y} = 2 + .5656x - .2028x^2$, we get $-\ddot{y} = 2.2178$. Whereas, by the method hereafter, it ſhould be $-\ddot{y} = 2.22956$; which therefore differs by .0117.

PROBLEM.

190. *Let the angle of elevation be* 30 *degrees, and the projectile velocity the greateſt poſſible, to find the range* A K.

Here we have $a = .57735$, $s = 1.1547$, by the tables of ſines, $ss = \frac{4}{3}$, and $n = 1$, by ſuppoſition; hence $y = .57735x - \frac{2}{3}x^2 - .5132x^3 - .168x^4 + .0745x^5$. Now if we make $y = o$, and divide by .57735, we get $1 = 1.1547x + .8889x^2 + .29098x^3 - .129x^4$; whence we get 1 : 1.1547 : 2.2222 : 3.8833 : 6.6664 : 11.6472 : 20.2181 : 35.1379 : 61.0748 : 106.1377. and $x = .57543$ nearly

nearly. For the values of x and its powers wrote into the equation give $1 = 1.00008$.

If this number be multiplied by 21214, the value of h, we get 4069 yards for the horizontal range A K.

PROBLEM.

191. *Let the angle of elevation be* 60 *degrees, and the reſt as before, to find the horizontal range* A K.

By the tables of ſines we have $a = 1.732$, $s = 2$, and as $n = 1$ by ſuppoſition, we get $b = 2$, $c = \frac{8}{3}$, $d = 1.512$, $-f = 1.428$; and therefore $y = 1.732x - 2x^2 - \frac{8}{3}x^3 - 1.512x^4 + 1.428x^5$. Now if $y = o$, and we divide by 1.732, we get $1 = 1.1547x + 1.5396x^2 + .873x^3 - .8245x^4$. The root being found in the uſual manner gives $x = .49323$, very nearly: for the values of x and its powers wrote into the equation give $1 = 1.000003$. This number, multiplied by 21214 feet, gives 3487.7 yards for the horizontal range, ſhort by 581 yards to that when the angle of elevation is 30 degrees.

PROBLEM.

192. *Let the inclination of the plane* A N, *either above or below the horizon, be given, to find the point* N, *where the ſhot will hit that plane.*

Suppoſe p the tangent of the angle N A Q, then will Q N $= y = px$, and this value of y wrote into the equation of *Art.* 175, with a poſitive ſign when the point N is above the horizon, or negative when below, gives $\pm p = a + bx + cx^2 + dx^3 + fx^4$; and if $p = .1$, $n = 1$, $ss = 2$, we get $.9 = x + .9428x^2 + .43095x^3 - .10907x^4$. If both ſides are divided by .9, we ſhall have $1 = 1.1111x + 1.0475x^2 + .4788x^3 - .1212x^4$. Hence 1 : 1.1111 : 2.282 : 4.1782 : 7.4436 : 13.6262 : 24.6707 : 44.7425 : 81.178 : 147.2255, and $x =$

.55138

.55138 nearly. For the values of x and its powers wrote into the equation give $1 = 1.00016$.

If the object is below the horizon and the reſt as before, then $-p = .1$, and the equation will then be $1 = .90909x + .85709x^2 + .39177x^3 - .0991x^4$, and $1 : .90909 : 1.68353 : 2.70142 : 4.15596 : 6.6631 : 10.51112 : 16.62728 : 26.32379 : 41.64033 : 65.89056 : 104.1574 : 164.871 : 260.8211 : 412.70435 : 653.0166 : 1033.24188$, and therefore $x = .632$; for if the values of x and its powers are wrote into the equation, give $1 = 1.000000$.

PROBLEM.

193. *The angle of elevation and the horizontal range being given, to find the ratio n of the greateſt velocity to the projectile one.*

If we ſuppoſe $y = o$, in *Art.* 175, we get $a = bx + cx^2 + dx^3 + fx^4$; and $2b = ss$, $3c = ns^3$, $12d = 2nns^4 - ans^3$, $60f = 4n^3s^5 - 8an^2s^4 + ns^3$; theſe values, being wrote into the equation, give

$$a = \tfrac{1}{2}ssx + \tfrac{1}{3}ns^1x^2 + \tfrac{1}{6}nns^4x^3 + \tfrac{1}{15}n^3s^5x^4 - \tfrac{1}{12}ans^3x^3 - \tfrac{2}{15}annx^4 + \tfrac{1}{60}ns^3x^4$$

Now when the angle of elevation and the range x are given, the value of n may be found in the uſual manner.

EXAMPLE.

194. *Let the angle of elevation be* 45 *degrees.*

Then as $a = 1$, $ss = 2$: we get

$$1 = x + .9428nx^2 + \tfrac{2}{3}nnx^3 + .37712n^3x^4 - .2357nx^3 - \tfrac{8}{15}nnx^4 + .04714nx^4.$$

Now if the range $x = .59262$, then $x^2 = .351198$, $x^3 = .208127$, $x^4 = .12334$; and theſe wrote into the

the equation give $1 = .59262 + .28787n + .07297n^2 + .04651n^3$; which gives $n = 1$ nearly: for we get $1 = .99996$. wanting 00004 only.

EXAMPLE.

195. Let the angle of elevation be 45 degrees, and the horizontal range $x = .7405$; then as $x^2 = .54834$, $x^3 = .40604$, $x^4 = .30067$. Theſe values wrote into the equation of the laſt article give $1 = .7405 + .43544n + .11034n^2 + .113388n^3$, whoſe root is $n = .5$; for $n^2 = .25$, $n^3 = .125$, $n^4 = .0625$; and theſe values wrote into the equation give $1 = 1.0000$.

Theſe two examples agree exactly with the *Art.* 179, 185, as it muſt happen; but when the value of n cannot be gueſs'd at, the value of x muſt be ſubtracted from unity, and both ſides divided by the difference, then the values of n may be found by the general rule for finding the roots.

PROBLEM.

146. *The ratio n of the velocities and the horizontal range being given, to find the ſine of the angle of elevation.*

If in *Art.* 175, we make $y = o$, the equation there found becomes $a = bx + cx^2 + dx^3 + fx^4$, and $b = \frac{1}{2}.ss$, $c = \frac{1}{3}ns^3$, $d = \frac{2nns^4 - anz}{12}$, $f = \frac{n^3s^5}{15}$, &c.

Now if z denotes the ſine and u the coſine of the angle of elevation, then will $a = \frac{z}{u}$, $s = \frac{1}{u}$, by the property of the circle; and hence $b = \frac{1}{2uu}$, $c = \frac{n}{3u^3}$, $d = \frac{2nn - nz}{12u^4}$, and $f = \frac{n^3}{15u^5}$. Theſe values being wrote into the equation, and the whole multiplied by u, give $z = \frac{x}{2u} + \frac{nxx}{3uu} + \frac{2nn - nz}{12u^3}x^3 + \frac{n^3x^4}{15u^4}$.

Whence, if we ſuppoſe $x = az + bz^2 + cz^3 + dz^4$, and we write the value of x and thoſe of its powers, we get $x = 2uz - \frac{8nu2^2}{3} + \frac{40nnuz^3}{9} - \frac{1088n^3uz^4}{135} + \frac{4nuz^4}{3}$. But if a denotes the ſine of the angle double that of elevation, then will $a = 2uz$, and $a = 2z - z^3 -$ &c. by the property of the circle; by writing theſe values into the laſt equation it becomes $x = a - \frac{2naa}{3} + \frac{5nna^3}{9} - \frac{68n^3a^4}{135}$. From hence we get $a = x + \frac{2nx^2}{3} + \frac{nnx^3}{3} + \frac{2n^3x^4}{15}$.

N. B. This ſeries is not exact, and gives the ſines only nearly, as the following examples will ſhew; thus, if $n = 1$, and $x = .59262$, then will $a = .91257$, which ſhould be unity by *Art.* 179; the difference being .08743.

Again, let $n = 1$, and $x = .57543$, then will $a = .8660$; but ſhould be 8743 by *Art.* 190; and therefore the difference is .0083 only.

With all the ſkill I may have in infinite ſeries, I could not make this to anſwer exactly; it would require two terms more at leaſt, which makes the calculation ſo very operoſe and intricate, that it is hardly worth while to attempt it: beſides in practice too much exactneſs is not required.

Second METHOD *of ſolving the ſame Problem by the differences.*

PROBLEM.

197. *The ratio of the greateſt and projectile velocities being given, together with the angle of elevation, to find the velocity at the higheſt point* D.

Becauſe

Becauſe $q\frac{2z}{r}=h\ddot{y}$ by *Art.* 177, and $\dot{z}=\sqrt{1+\dot{y}^2}$; now if the firſt ſide be multiplied by $\dot{z}$, and the ſecond by its equal $\sqrt{1+\dot{y}2}$, we get $\dot{z}q\frac{2z}{r}=h\ddot{y}\sqrt{1+\dot{y}^2}$; and if $\frac{1}{2}m$ denotes the fluent of $\ddot{y}\sqrt{1+\dot{y}^2}$; we ſhall have $rq\frac{2z}{r}=2A+hm$ for the fluent by the remark we made after *Art.* 80; but at the vertex D, where $z=\dot{y}=0$, and $q\frac{2z}{r}=1$, we get $\frac{r}{2}=A$, and therefore $rq\frac{2z}{r}=r+hm$, or $q\frac{2z}{r}=1+\frac{hm}{r}$; this value being wrote into the ſquare $vvq\frac{2z}{r}=h\dot{z}^2$, gives $vv\times 1+\frac{hm}{r}=h\dot{z}^2$. From whence it appears that the ratio of the velocity at the vertex, and any other point where the angle made by the tangent and abſciſſa is known, is given.

COR.

198. Hence, we may deduce the values of $\dot{x}$ and $\dot{y}$; for if we write the value $1+\frac{hm}{r}$ of $q\frac{2z}{r}$ into $q\frac{2z}{r}=h\ddot{y}$, we get $1+\frac{hm}{r}=h\ddot{y}$, or $1=\frac{h\ddot{y}}{r+hm}$, or $\dot{x}=\frac{rh\ddot{y}}{r+hm}$, or elſe multiplying both ſides by $\dot{y}$, $\dot{y}=\frac{rh\dot{y}\ddot{y}}{r+hm}$.

It is to be obſerved that the equations juſt found are for the deſcent from the higheſt point D; but when the body aſcends, then m becomes negative;

consequently $vv \times 1 - \frac{hm}{r} = h\dot{z}^2$, $\dot{x} = \frac{rh\ddot{y}}{r-hm}$ and $\dot{y} = \frac{rh\dot{y}\ddot{y}}{r-hm}$ in this case.

COR.

199. Since we have $\dot{z} = v\dot{\imath}$ and $v\dot{x}q\frac{z}{r} = c\dot{z}$, by *Art.* 177, we get $\dot{x}q\frac{z}{r} = c\dot{\imath}$; or because $q\frac{z}{r} = \sqrt{1 - \frac{mh}{r}}$ and $\dot{x} = \frac{r\dot{y}}{r-hm}$, we get $c\dot{\imath} = \frac{r^{\frac{1}{2}}\dot{y}}{\sqrt{r-mh}}$.

EXAMPLE.

200. *Let the angle of elevation be* 45 *degrees, and the projectile velocity at the point* A *equal to the greatest possible, to find the velocity* c *at the highest point.*

Then $vv \times \sqrt{r + hm} = hr\dot{z}^2$ gives $h = \frac{vv}{\dot{z}^2 + nm}$; because m is here negative and $\frac{vv}{r} = n$. Now as $\dot{z}$ expresses the secant at A, we have $\dot{z} = 2$, and $n = 1$, whence $h = \frac{vv}{2+m}$; but the fluent of $\ddot{y}\sqrt{1+\dot{y}^2}$ gives $m = \dot{z}\dot{y} + log.\ \dot{y} + \dot{z}$, because $\dot{z} = \sqrt{1+\dot{y}^2}$; or because $\dot{y} = 1$, $m = \dot{z} + log.\ 1 + \dot{z} = 2.2956$; we have $h = \frac{vv}{4.2956}$. Hence, in an 18 pounder, $vv = 21214$ feet, by *Art.* 182; consequently, $h = 4938.6$ feet.

Before we apply this, to find the values of $\dot{x}$ and $\dot{y}$, it will save much trouble to insert the following table, where the values of m are under that letter, according to those supposed of $\dot{y}$ under it against each other.

other. The uſe of this table will appear in the following

EXAMPLE.

201. *To find the abſciſſa* A B *correſponding to the greateſt ordinate* B D.

Becauſe $b = \frac{vv}{4.29559}$, or as $vv = r$ by ſuppoſition $b = \frac{r}{4.29559}$; and if this value be wrote into $\dot{x} = \frac{rb\ddot{y}}{r - hm}$, we get $\dot{x} = \frac{r\ddot{y}}{4.29559 - m}$.

$\dot{y}$	m	$\dot{y}$	m	$\dot{y}$	m	$\dot{y}$	m
1	2.29559	$\frac{1}{2}$	1.04027	$\frac{3}{5}$	1.26642	$\frac{1}{8}$	0.25077
$\frac{1}{3}$	0.67872	$\frac{3}{4}$	1.63064	$\frac{4}{5}$	1.75720	$\frac{3}{8}$	0.76722
$\frac{2}{3}$	1.42638	$\frac{1}{5}$	0.40265	$\frac{1}{6}$	0.33500	$\frac{5}{8}$	1.32727
$\frac{1}{4}$	0.50688	$\frac{2}{5}$	0.82091	$\frac{5}{6}$	1.84322	$\frac{7}{8}$	1.95340

Now if the ſpace of which $\dot{x}$ is the fluxion be divided into ſix equal parts by 7 ordinates, then the parts of the baſe $\dot{y}$ will be $0 : \frac{1}{6} : \frac{2}{6} : \frac{3}{6} : \frac{4}{6} : \frac{5}{6} : \frac{6}{6}$, or $0 : \frac{1}{6} : \frac{1}{3} : \frac{1}{2} : \frac{2}{3} : \frac{5}{6} : 1$; by ſubtracting the correſponding values of m from 4.29559, and dividing unity by the difference, we get .23279 : .25248 : .27648 : .30719 : .34853 : .40776 : .5. Now making the ſum of the firſt and laſt equal to A, that of the ſecond and laſt but one equal to B, that of the third and laſt but two C, and the fourth D, then A = .73279, B = .66024, C = .62501, D = .30719; and writing theſe values into the form $\frac{41A + 216B + 27C + 272D}{840}$, found in *Art.* 76, for ſeven ordinates, and dividing the Sum 273.0872, by the denominator 840, give $x = .32511$ for the number

number required; which exceeds .32456, found in *Art.* 180, by .00055 parts.

EXAMPLE.

202. *To find the greateſt ordinate* B D.

Since $\dot{y} = \frac{r\dot{y}\ddot{y}}{4.29559 - m}$, from whence it appears, that if we multiply the ordinates found above each by the reſpective values of $\dot{y}$, we get 0 : .04288 : .09216 : .15359 : .23233 : .33980 : .5. Hence A = .5, B = .38188, C = .32451, D = .15359. Theſe values wrote into the form above, and the ſum 153.5243, divided by 840, give y = .1827; that found in *Art.* 181, is .1826.

EXAMPLE.

To find the part B K *of the range.*

In order find this part, the tangent of the angle at K muſt be known firſt; to effect which, let *ab* be drawn parallel to the baſe BK, ſo as the tangent y at b be unity; then by means of the equations $\dot{x} = \frac{r\ddot{y}}{4.29559 + m}$, $\dot{y} = \frac{r\dot{y}\ddot{y}}{4.29559 + m}$, we are to find D*a* and *ab*: now if we ſuppoſe the baſe $\dot{y}$ divided into five ordinates, then will 0 : $\frac{1}{4}$: $\frac{1}{2}$: $\frac{3}{4}$: 1 : be the values of $\dot{y}$; and dividing unity by the ſum of 4.29559, and the values of m, correſponding to thoſe of $\dot{y}$, we get .23279 : .20822 : .18741 : .16874 : .15171; and hence A = .3845, B = .37696, C = .18741. Theſe values wrote in the general form $\frac{7A + 32B + 12C}{90}$, give *ab* = .188923.

Now if each of theſe ordinates be multiplied by the reſpective values 0, $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, 1, we get 0 : .05205 : .0937 : .12655 : .15171, hence A = .15171, B = .1786, C = .0937; theſe values, when

when wrote into the general form above of five ordinates, give $y = aD = .08779$.

Now the queſtion is to find ſuch a value for $\dot{y}$, that the ſpace found by means of the equation $y = \frac{r\dot{y}\ddot{y}}{4.29559+m}$, may be equal to B*a*; for this value being once obtained, it need only be added to unity, which is that at *b*, to have that at K, by which the difference between BK and *ab* may be found.

We have found $BD = y = .1827$, in *Art.* 202, and $aD = .08779$ juſt now, whence $Ba = .0949$; which being divided by the laſt ordinate .15171, found above, gives .6256, which muſt be greater than the value of $\dot{y}$ required, which we ſhall ſuppoſe .47; whence if the ſpace .47 be divided into two equal parts .235, then 1 : 1.235 : 1.47 will be the parts correſponding of the baſe: then the values of *m*, when $y = 1.235$, and 1.47, will be 2.0005, 3.79, which being added to 4.29559, and unity divided by the ſum, give .15171 : .15898 : .12376 : and theſe values wrote into the general form $\frac{A+3B}{6}z$, for three ordinates, give .1519; which, multiplied by the baſe .47, gives $Ba = x = .07139$, which being leſs than .0949, ſhews the value 1.47 for $\dot{y}$ is too ſmall, and hence a greater muſt be aſſumed; and if we add this to 816, we get $BK = .2603$ inſtead of .2680, ſo that the difference is .0077, ſo ſmall it would be needleſs to extend the explanation of this method any farther. The reader muſt be ſenſible, that the latter part is attended with great difficulties, the trials to be made for finding the tangent of the angle, in which the body ſtrikes the ground, are ſo very tedious and operoſe, that this method will hardly ever be uſed, at leaſt the latter part; as to the firſt,

it

it may be applied to good purpoſes, and the reſt ſupplied by what we ſhall ſay in the

THIRD METHOD.

PROBLEM.

204. *The projectile velocity at the vertex* D, *being given, to find the abſciſſa* AB *correſponding to the greateſt ordinate* BD.

We have $y = xx + cx^3 + dx^4 + fx^5 + gx^6 + kx^7$, by *Art.* 177; and $b = \frac{1}{2}$, $3c = n$, $6d = nn$, $60f = 4n^3 + n$, $180g = 4n^4 + 7nn$, $315k = 2n^5 + 16n^3 - \frac{3}{8}n$. It is to be obſerved, that this ſeries was continued further than the former, becauſe all the terms, which contain the odd powers of x, become negative in the aſcent of the body, and n expreſſes the ratio of the greateſt and projectile velocity at the vertex D.

Now ſince $b = \frac{vv}{4 \cdot 29557}$, by *Art.* 200; and $\frac{b}{r} = \frac{1}{4 \cdot 29559} = .23273$, becauſe $r = vv$ by ſuppoſition, and $n^2 = .05419$, $n^3 = .01261$, $n^4 = .00293$; $n^5 = .00068$. Hence $b = \frac{1}{2}$, $c = .07759$, $d = .00903$, $f = .00472$, $g = .00217$, $k = .00037$; theſe values wrote into the equation before give $y = \frac{1}{2}x^2 + .07759x^3 + .00903x^4 + .00472x^5 + .00217x^6 + .00037x^7$: and the fluxion of this equation will be $\dot{y} = \dot{x} + .23279x^2 + .03613x^3 + .0236x^4 + .01304x^5 + .00194x^6$. By means of theſe two equations all the ſeveral parts of the projectiles may be determined; but as the value of y is greater than unity, the ſeries will not converge without that of x being diminiſhed, which is done in the ſhorteſt manner by writing $2x$ inſtead of x; we ſhall then have $\dot{y} = 2\dot{x} + .9312x^2 + .289x^3 + .3776x^4 + .4173x^5 + .1242x^6$: by this means the value of x is diminiſhed

diminiſhed to one half. It muſt be obſerved that every other term in theſe ſeries become negative in the aſcent.

EXAMPLE.

205. *The angle of elevation at* A *being in the caſe above* 45 *degrees.*

The fluxion $\dot{y}$ being the tangent when $\dot{x}$ is the radius, and therefore $\dot{y} = \dot{x} = 1$, by ſuppoſition, we get $1 = 2x - .9312x^2 + .289x^3 - .3776x^4 + .4173x^5 - .1242x^6$, in the aſcent; and by the method of finding the roots we have $x = .697$; for this value wrote into the equation gives $1 = 1.00513$.

Now $2x = 1.394 =$ AB, when multiplied by h, and as $h = 4938.6$ feet by *Art.* 200, that abſciſſa will therefore be 6884.4 feet; and as we have found 6885 feet in *Art.* 182 for that diſtance, the difference is only .6 of a foot. Now, to find the greateſt ordinate BD, we muſt write the value 1.394 of x, and that of its powers, into $y = \frac{1}{2}x^2 - .07759x^3 + .00903x^4 - .0047x^5 + .00217x^6 - .00037x^7$, which give $y = .78283$; and this value, multiplied by $h = 4938.6$, gives BD $= 3866$ feet; which is leſs by 7.6 feet than that found in *Art.* 182.

EXAMPLE.

206. *Let it be required to find the remainder* BK *of the horizontal range.*

Becauſe $y = .5x^2 + .07759x^3 + .00903x^4 + .00472x^5 + .00217x^6$, and $y = .78282$; if we divide by this value of y we get $1 = .638708x^2 + .0991x^3 + .0115x^4 + .006x^5 + .002776x^6$: the ſquare root of this equation is $1 = .7992x + .062x^2 + .0024x^3 + .0007x^4$; whoſe root is $x = 1.1514$. For by writing this value of x, and thoſe of its powers into the equation, give $1 = 1.00728$ nearly; which ſhews that this value is ſome very ſmall matter more than the real one.

If this value 1.1514 of x be multiplied by 4938.6 feet, that of h, we get 5686.3 feet for BK; and as we have found 5687 feet for that diſtance, in *Art.* 182, the difference is not above one foot.

It is plain that $y = x + .2327x^2 + .0961x^3 + .0276x^4$ expreſſes the tangent of the angle in which the ſhot ſtrikes the ground, and as $y = 1648$, that angle will be 58 degrees and 45 minutes, which exceeds that found in *Art.* 184 by about two degrees.

We have thus ſhewn how the ſeveral problems of this kind may be ſolved by three different methods, all ſufficiently accurate for practice, wherein no mathematical exactneſs is required more than to come as near as it can be done upon the ground; for as the batteries and the object are ſeldom upon an exact level, the gunner goes by gueſs only, and when the object is either much higher or lower, the difference is only judged at by the eye: and when the ſhot or ſhell falls ſhort or exceeds the mark, the powder is increaſed in one caſe and diminiſhed in the other.

There are many other problems given by authors; ſuch as to find the ratio of the reſiſtance to the force of gravity, ſo that the body may deſcribe a given curve, or to find the curve deſcribed by the body, when the reſiſtance is to its gravity in any other ratio than that of the ſquares of the velocities; but as there are no other laws in nature, as far as I can find, and ſerve only to puzzle the reader, and to loſe his time in uſeleſs inquiries, we have not mentioned them, and refer ſuch readers as are fond of needleſs ſpeculations to authors who wrote profeſſedly on them.

SECTION

SECTION VIII.

The true Figure of the Earth determined from actual Meaſures.

SINCE Sir Iſaac Newton has firſt determined the figure of the earth, in his Principles of Natural Philoſophy, moſt mathematicians of the firſt claſs have made it their principal ſtudy. Many works have ſince appeared upon this ſubject without agreeing upon the ratio of the axis to the diameter of the equator, notwithſtanding the meaſures of a meridian degree made ſince in different latitudes, and the great number of experiments made upon the length of pendulums vibrating ſeconds in different parts of the world.

Sir Iſaac conſidering the earth as a fluid body, and upon this ſuppoſition, that any meridian ſection muſt be an ellipſis, otherwiſe the particles of matter could not remain at reſt amongſt each other in the revolution of the earth about its axis, and beſides that there can be no other figure in nature which has that property; from thence, and the meaſure of a meridian degree in France, found that the ratio of the axis to the diameter of the equator was as 229 to 230.

But finding afterwards that, by ſome experiments on pendulums made in the Weſt Indies, this ratio muſt be ſomething greater; and yet moſt authors make it leſs, as Maclaurin, Thomas Simpſon, and Clairaut; as to Bouguer, he makes it much greater than it ſhould be.

It is for this reaſon, I have attempted to find this ratio more accurately, by means of ſome properties of the ellipſis, and the two beſt meaſures of a meridian degree, near the pole and the equator, and

found it to be as 215 to 216. That this ratio is the neareſt poſſible, expreſſed by three figures, appears by its agreeing as nearly as can be with the actual meaſures and the experiments made on the length of pendulums vibrating ſeconds in different latitudes.

Had authors compared their theories with the actual meaſures and experiments, they would have found, that their ratios were not ſufficiently exact; which is the beſt and moſt certain method in philoſophical inquiries. The French have ſpared no pains or expence to determine this queſtion ſo uſeful in geography and navigation, and though they have ſucceeded beyond expectation in their meaſures near the pole and at the equator, the caſe is quite different with thoſe in France; for there are no leſs than five different meaſures of the ſame degree, &c. 57060, 57292, 57183, 57074, and 56926 toiſes: ſometimes it is ſaid to be the meaſure of a meridian degree at Paris, and at others the mean meridian of France. In ſhort, theſe meaſures are ſo uncertain, that it is not ſafe to apply them to any particular meridian, for want of knowing where they begun or where they ended; for Clairaut, Maupertius, and Bouguer do not mention any thing about it in their works.

In what follows we ſhall ſuppoſe, with other authors, that all celeſtial bodies attract each other in the ratio of their quantities of matter directly, and the ſquares of their diſtances inverſly: this Sir Iſaac has proved to be ſo in all the planets, and from thence that this law was univeral.

THEOREM. Fig. 57.

207. *A particle of matter placed on the ſurface of a ſphere* A M Q D *is attracted towards the Center* C, *by a force proportionally to the radius* C M.

For

For the quantity of matter in a ſphere of an uniform denſity, is as the cube of its radius; and ſince the attraction is as the quantity of matter directly, and the ſquare of the diſtance inverſly, the attraction of a particle of matter placed on the ſurface of a ſphere is therefore, as the cube of the radius directly and the ſquare inverſly; that is, as the radius directly.

This has likewiſe been proved by Sir Iſaac, in *Prop.* 72. *Book* I.

COR.

208. Hence, if the attraction ſhould ceaſe at a certain diſtance CI from the center, or be deſtroyed by another force in an oppoſite direction, it is evident, that there would be a hollow within of a concentric ſphere.

That there is a repelling force in nature, is confirmed by many experiments; but whether it reaches ſo far as the center of bodies, is not certain: it is true the attractive force ſeems to reach ſo far, and by a parity of reaſon the repelling force may do the ſame.

THEOREM. Fig. 58.

209. *If a ſphere, whoſe particles are freely diſpoſed, turns round one of the diameters* A a *as an axis, with ſuch a force as may be compared to that of attraction, the ſphere will be changed into a ſpheroid.*

Let PM be perpendicular to the axis A a, and reduce the force of attraction, in the direction CM, into the perpendicular PM and the parallel CP, to the axis A C; then the force of attraction at M, in the direction MP, will be to the force of attraction at D, in the direction DC, as PM is to CD; and ſince the centrifugal forces in different circles, in equal times, are as their radii, *Art.* 114, the centrifugal force at M is to the centrifugal force at D,

are

are likewiſe as PM to CD. Now as the centrifugal force acts in a contrary direction to that of gravity, it will diminiſh it in proportion as it approaches to the equator CD: whereby the ſphere will be protracted at the poles, and riſe towards the equator, otherwiſe the particles of matter could not remain in a ſtate of reſt.

Let MN be to DB as the centrifugal force at M to the centrifugal force at D, then will PM : CD :: MN : DB, and by compoſition PM : CD : : PN : CB. Which is the property of the ellipſis.

COR.

210. It is manifeſt that the ſpheriod becomes proeminent at the equator in proportion as the centrifugal force increaſes, or, which is the ſame, as the periodic time decreaſes, till the centrifugal force becomes equal to that of gravity, and then the particles of matter would remain ſuſpended between the two equal forces; but if the centrifugal force ſhould prevail the leaſt, they would fly off without end.

COR.

211. It is no leſs evident, that if the particles of matter were of different denſities, it would not change the nature of the figure. For in the formation of the body, the particles would naturally diſpoſe themſelves ſo as to be at reſt with each other, and the heavieſt would by their greater force approach the nearer to the center; and no other change could happen, than that the ſpheroid would be denſer near the center than at the ſurface: the centrifugal force muſt produce the ſame effect in both caſes.

COR.

212. If there was an empty ſpace about the center of the ſphere at reſt, when that ſphere is changed into a ſpheroid by its revolution about its axis, the hollow ſphere will at the ſame time be changed into a ſphe-

a ſpheroid ſimilar to the outward one. For the centrifugal force, as well as that of attraction, will act in the ſame manner upon every particle of matter, whether the ſpheroid was hollow within or not.

REMARK.

It is highly probable that all the planets are hollow within, from this general maxim in philoſophy, that nothing has been created in vain, and more is in vain when leſs will do; and ſince after a certain thickneſs of matter near the ſurface, there ſeems to be no more neceſſary, at leaſt as far as we can judge; yet as men can reaſon but from appearances, and they are in favour of an emptineſs within the planets, there is no abſurdity to ſuppoſe they are ſo in effect. It appears likewiſe that all bodies formed by the mutual attraction of the particles will aſſume the form a ſphere, if no impediment intervenes and they do not turn about an axis, this appears from drops of water and quick-ſilver: but if theſe bodies revolve about an axis, they will become ſpheroids, more elevated at the equator than at the poles: this is evident in fluids, and even when ſome parts are only ſo, as has been found by obſervation in all the planets, although it is not certain whether they contain any fluid matter or not.

THEOREM.

213. *The direction of gravity on the ſurface of planets is every where perpendicular to the ſurface.*

Experience ſhews it to be ſo in regard to the earth, and it is manifeſt from the laws of hydroſtatics, that the direction of gravity in all fluids muſt be perpendicular to the ſurface, otherwiſe the particles could never be at reſt amongſt each other: it is no leſs evident, that the direction of gravity is the ſame every where elſe, ſince bodies could not remain at reſt after their fall. For ſuppoſe A M, *Fig.* 54,

to be a part of the ſurface, CM the direction of gravity; then it is manifeſt, that if the angle CMT, made by the direction CM and the tangent MT to the ſurface at M, be leſs than a right angle, the body, inſtead of remaining at reſt after the fall, would move in the direction MT, which is contrary to experience. As all planets revolve about their axes, and the attraction acts upon them in the ſame manner, and by the ſame law, as upon the ſurface of the earth, it muſt evidently follow that the ſame cauſes produce the ſame effect.

N. B. Several authors endeavour to find the direction of gravity by a long and tedious inveſtigation, without having regard to the laws of hydroſtatics, and ſome pretend, that it deviates from the perpendicular; this ariſes from their ſuppoſing the ſpheroid to be at reſt: and as this is contrary to fact, their concluſion is evidently erroneous.

THEOREM. Fig. 59.

214. *The attraction towards the center of a particle, placed on the ſurface of a planet, is as its diſtance from the center.*

Becauſe three powers or forces remain in a ſtate of reſt, when they are to each other as the reſpective ſides of a triangle formed by their directions: if Aa be the axis, CB the radius of the equator, MK perpendicular to the tangent at M, and CM a ſemidiameter; then becauſe MK is the direction of gravity at M by the laſt, CK that of the centrifugal force at that point, and CM that in which the attraction tends towards the center: therefore CK is to KM as the centrifugal force is to the force of gravity, and KM is to CM as the force of gravity is to its part by which it tends towards the center. This has been proved by Sir Iſaac in the 3d *Cor.* after *Prop.* 91, *Book* I, from different principles.

COR.

COR.

215. Hence, if MK be produced till it meets the conjugate diameter CN to CM in L; then if the centrifugal force, expreſſed by CK, be reduced into two others CL, LK, the firſt perpendicular and the ſecond parallel to the direction MK of gravity, ML will expreſs the total force of gravity, and MK the part by which it acts on a particle placed at M, after being diminiſhed by LK or the centrifugal force. This has likewiſe been proved by Clairaut, p. 188. If the gravity in any given latitude be known, the total gravity ML and the centrifugal force CK may be found.

COR.

216. When the point M coincides with the point B in the equator, the perpendicular MK becomes equal to the parameter of the ſemi-axis CB, by the property of the ellipſis: hence, if *a* expreſſes the force of gravity at B, then will CB : CA : : CA : *a*; and when the point M coincides with the point A, then MK becomes equal to CA; that is, if *p* expreſſes the gravity at the pole, then will CB : CA : : *p* : *a*; which ſhews, *that the force of gravity at the pole is to the force of gravity at the equator reciprocally as the radius of the equator is to the radius of the axis.* This is likewiſe proved from ſtatics.

COR.

217. If PM be drawn perpendicular to the equator CB; then becauſe the centrifugal force CK at M is to the centrifugal force at B, as CP is to CB, by the property of the ellipſis; or as the circumference deſcribed by the point M is to the circumference deſcribed by the point B. This agrees with what has been proved in *Art.* 114; and therefore confirms what has been proved in the *Articles* 213, 214, 215, and 216.

These are the only elementary propositions necessary to determine the true figure of the earth from observations, which being deduced from the nature of the subject, and some properties of the ellipsis, are so plain and easy as to leave not the least doubt of their certainty; which is not so with most authors, who make use of fluxions and infinite series: for if the least mistake happens, it is difficult to find it out in long and tedious algebraic computations, the deducing a wrong direction of gravity is a convincing instance, besides many others equally as absurd.

PROBLEM.

218. *The ratio between the actual gravities in two different latitudes being given, to find the ratio between the axes of the ellipsis.*

Let CB $= 1$, CA $= b$, CP $= u$, PM $= y$, and $1 - bb = n$; then will $yy = bb \times \overline{1 - uu}$ be the equation of the ellipsis, and PK $= bbu$; and as the sum of the squares of KP and PM is equal to the square of MK, we have $MK^2 = bb \times \overline{1 - uu} + bbuu$. But to find such a value for MK as may be applied to practice, let r be the sine and s the cosine of the latitude MKB, the radius being unity; then will PK (bbu) : PM (y) : : s : r, or $sy = bbur$, and writing the value of y into the equation $yy = bb \times \overline{1 - uu}$, we get $bbrruu = ss - uuss$, and this value of uu into that of MK^2, gives $MK^2 = \frac{b^4}{bbrr + ss}$, when reduced under the same denomination, and unity wrote for its equal $rr + ss$, or because $1 - bb = n$, $bb = MK \sqrt{1 - nrr}$.

COR.

219. Since, when the point M coincides with point B in the equator, then will $r = o$, and the last equation becomes $bb = MK$; hence, if p is to a as the

the gravity at any point M is to the gravity at the equator, we get $a : p :: bb : \frac{bb}{\sqrt{1-nrr}}$, or $a = p\sqrt{1-nrr}$; which shews, *that the force of gravity in any latitude* M *is to the force of gravity at the equator, as unity is to* $\sqrt{1-nrr}$, *when r denotes the sine of that latitude.* From whence the ratio of the axis to the diameter of the equator may be determined.

COR.

220. Since $\overline{1-nrr}^{1-\frac{1}{2}} = 1 + \frac{1}{2}nrr$, nearly, the last equation $a = p\sqrt{1-nrr}$ gives $\frac{1}{2}anrr = p - a$, and as $\frac{1}{2}an$ is constant in the same sphere, the difference $p - a$, between the gravities in any latitude M and at the equator, is as the square of the sine r of that latitude nearly. From whence follows the common rule given first by Sir Isaac, *that the increase of gravity from the equator towards the pole, is as the square of the sine of the latitude nearly.*

From this Corollary it appears, that the decrease of gravity near the equator is much greater than towards the poles, because the sines increase much faster at the beginning of the quadrant than near the end; for which reason, experiments made on gravity near the equator are much more uncertain than farther off; because the errors in point of latitude are more there than elsewhere: as pendulums which vibrate in equal times, and by which the ratio between gravities are discovered, their exact lengths are extremely difficult to be found, as will appear from those two in the latitudes of 51° 32′ and 53° 11′; for their difference is no more than .004 parts of an inch. It will therefore be more convenient to find that ratio by the measures of two meridian degrees in two different latitudes, as far distant from each other as can be; for which reason we shall use

thoſe meaſured at the equator and near the pole, which at the ſame time are the moſt exact of any we know of.

PROBLEM.

221. *From the given ratio between the force of gravity at Paris and at the equator, to find the ratio between the axis and the diameter of the equator.*

Mr. Clairaut ſays, in page 193, that a body deſcends through a ſpace of 2174.11 lines in a ſecond at Paris, and in the next page, through a ſpace of 2168.75 lines at the equator; theſe numbers divided by 5, in order to reduce them to a leſs denomination, gives $p = 43482$, and $a = 43375$. Now as Paris is in the latitude of 48° 50′, according to Clairaut, we have $r = .752798$, or $rr = .5667$ nearly; and theſe values wrote into the equation $\frac{1}{2}anrr = p - a$, *Art.* 220, give $n = \frac{1070}{1328033}$: from whence we get $1 - n = bb = \frac{1317333}{1328033}$, whoſe ſquare root is $b = \frac{51185}{52008}$, after having been divided by 7.

Hence, the neareſt ratio between the axis and diameter of the equator, expreſſed by three figures, is that of 248 to 247, according to theſe gravities; and not that of 230 to 231, as Maclaurin, Simpſon, and Clairaut would have it, by Clairaut's own principles.

PROBLEM.

222. *From the known meridian degrees in two different latitudes, to find the ratio of the axis to the diameter of the equator.*

Becauſe ſimilar circular arcs are as their radii, and the ſingle degrees of an ellipſis are ſenſibly equal to a degree of the circle deſcribed by the radius of curvature; the degree of a meridian at any point M is to the meridian degree at any other point as the radius of curvature at M is to the radius of curvature at the other point. But the radius of curvature at the point M is equal to the cube

cube of the perpendicular MK, divided by CA^4, *Art.* 42, and since $bb = MK\sqrt{1-nrr}$, *Art.* 218, the radius of curvature at M will be expressed by $bb \times \overline{1-nrr}^{-\frac{3}{2}}$; for the same reason, if s denotes the sine of any other latitude, $bb \times \overline{1-nss}^{-\frac{3}{2}}$, will express its radius of curvature.

COR.

223. Hence, as the sine s at the equator vanishes $bb \times \overline{1-nss}^{-\frac{3}{2}}$, becomes bb: therefore the meridian degree at the equator is to the meridian degree at M as bb is to $bb \times \overline{1-nrr}^{-\frac{3}{2}}$, or as $\overline{1-nrr}^{-\frac{3}{2}}$ to unity. Hence, the meridian degrees at the equator and in any latitude being given, the ratio of the axis to the diameter of the equator will be found, and on the contrary, the ratio of the axis to the diameter of the equator being given, that of the meridian degrees may be determined.

EXAMPLE.

224. Suppose $b = \frac{230}{231}$, according to Clairaut and others, then will $1 - \frac{bb}{} = n = \frac{461}{533611}$, and as the sine of the angle 66° 20′, which answers to the middle of the meridian degree measured by Maupertuis and others, is $r = .91589$, and its square $rr = .83885$; this value and that of n wrote into $\sqrt{1-nrr}$ give .99637, whose cube is .98914 nearly; and as the meridian degree, in the latitude 66° 20′, has been found to be 57838 toises, see page 162, Maupertuis, we have by *Art.* 223 this proportion 1 : 98914 :: 57438 : 56814 toises for the meridian degree at the equator, which exceeds the measured one 56753 by 61 toises.

If we suppose $b = \frac{229}{230}$ with Sir Isaac, we shall find 56811 toises, which exceeds the measured one by 58 toises: if we take 57422 toises for the meridian degree at the pole, as corrected by Bouguer, we shall find

find 56798 toiſes, which exceeds the meaſured one by 45 according to Clairaut, and 42 according to Newton.

PROBLEM.

225. *From the known meridian degrees at the equator and near the pole, to find the ratio of the axis to the diameter of the equator.*

Becauſe $\overline{1 - nrr}^{\frac{3}{2}} = 1 - \frac{3}{2}nrr$ nearly by the binomial rule, we have by the proportion in *Art.* 223, 57838 : 56753 :: 1 : $1 - \frac{3}{2}nrr$, and becauſe $rr = .83885$, by *Art.* 224, we get $n = \frac{1370}{144545}$ and $1 - n = bb = \frac{143175}{144545}$, or dividing by 5, $bb = \frac{28635}{28909}$, whoſe ſquare root is $b = \frac{169218}{170026}$.

The neareſt ratio expreſſed by three figures is greater than that of 211 to 210, and leſs than that of 632 to 629; but nearer to the laſt. This is true upon the ſuppoſition, that the meaſures of the meridian degrees are exact; but as this ratio does not agree with the obſervations on pendulums, there muſt be ſome ſmall error or other: if the correction mentioned by Bouguer, page 290, be admitted, which makes the degree near the pole 57422 toiſes only, we ſhall find in the ſame manner as before, that $n = \frac{13385}{144505}$ and $bb = \frac{143167}{144505}$, whoſe ſquare root gives $b = \frac{378374}{380138}$.

Hence the neareſt ratio between the axis is leſs than that of 214 to 215, and greater than that of 215 to 216; but neareſt to the laſt.

That this ratio is the neareſt to the true one, of all thoſe found by other authors, appears from its exact agreement with all the meaſures of a meridian degree in different latitudes, and with all the beſt experiments made on the lengths of pendulums vibrating ſeconds, as the following examples will ſhew.

EXAMPLE.

EXAMPLE.

226. *The meridian degree at the equator being given, to find that near Pellow in the latitude of* 66° 20′.

Since $b = \frac{215}{216}$ by the laſt, or $bb = \frac{46225}{46656}$, we get $n = \frac{431}{46656}$; and becauſe $rr = .83885$, by *Art.* 224, we get $\sqrt{1 - nrr} = .99611$, whoſe cube is .98837, and by the proportion in *Art.* 223, we have .98837 : 1 :: 56753 : 57421 toiſes for the meridian degree at Pellow, which wants but one toiſe of the corrected one.

EXAMPLE.

227. *The meridian degree at the equator being given, to find that at Paris in the latitude of* 48° 50′.

We have $rr = .5667$, by *Art.* 221, and as $n = \frac{431}{46656}$, by the laſt, we get $\sqrt{1 - nrr} = .99738$, whoſe cube is .99216, and by the proportion in *Art.* 224, we have .99216 : 1 : : 56753 : 57201 toiſes for the meridian degree at Paris, which exceeds the corrected one 57183 by 18 toiſes.

That the length of the meridian degree at Paris, we have juſt now found, is pretty exact, appears from the meaſure of gravity at Pellow and at Paris, made by Maupertuis, which he ſays is as 1.0014 to unity, page 226, Engliſh edit. As Pellow is in the latitude of 66° 48′, whoſe ſine r is $= .91913$, and $rr = .8448$ nearly, hence we get $\sqrt{1 - nrr} = .996$ nearly, the cube of which is .988, and by the proportion in *Art.* 223, .988 : 1 : : 56753 : 57442 toiſes for the degree in that latitude. And as the meridian degrees are as the cubes of gravity, *Art.* 222, the cube 1.0042 of 10014 is to unity, as 57442 is to 57201 toiſes, for the meridian degree at Paris, which is exactly the ſame as that computed before; conſequently, the true meridian degree of Paris

Paris cannot differ from that here computed more than in the laſt figure.

EXAMPLE.

228. *The meridian degree of the equator being given, to find that at London in the latitude of* 51° 32′.

The ſine of this latitude is $r = .78297$, and $rr = .61304$ nearly, and hence $\sqrt{1 - nrr} = .997164$, whoſe cube is .99152; and therefore .99152 : 1 :: 56753 : 57238 toiſes for the meridian degree at London.

The meridian degree meaſured by Mr. Norwood from London to York is different from that here found; for the difference between theſe places is 2° 28′, half of which, added to 51° 32′ of London, gives 52° : 46′ for the middle latitude of that diſtance, whoſe ſine is $r = .796178$ and $rr = .6339$ nearly; hence $\sqrt{1 - nnr} = .99707$, whoſe cube is .99123; and therefore .99123 : 1 :: 56753 : 57255 toiſes for the mean meridian, which is leſs than 57300, that found by Norwood, by 45 toiſes; and as this difference is greater than any error that could proceed from any inaccuracy of computation, it muſt have been an error of meaſurement: that this is the caſe appears from the next example, where the length of a degree is leſs, though 4° 15′ more northerly.

EXAMPLE.

229. *The length of a meridian degree at the equator being given, to find that at Edinburgh, in the latitude of* 55° 47′.

The ſine of that latitude is $r = .82692$, and $rr = .68379$, and hence $\sqrt{1 - nrr} = .99684$, whoſe cube is .99055, and therefore .99055 : 1 :: 56753 : 57294 toiſes, for the meridian degree required.

EXAMPLE.

EXAMPLE.

230. *The meridian degree at the equator being given, to find that at Dublin in the latitude of* 53° 11′.

The sine is here $r = .800557$, and $rr = .64089$, and so $\sqrt{1 - nrr} = .99703$, whose cube is .99112; and therefore .997112 : 1 :: 56753 : 57261 toises for the meridian required.

Having shewn how to find the meridian degree in any given latitude, in a more concise and exact manner than any author has done before, which is very useful in correcting geography, and in navigation, it remains now to shew how to find the length of pendulums vibrating seconds in any latitude, in as correct a manner as can be expected, and by which it will be confirmed how exact the ratio of the axis to the diameter of the equator has been determined.

PROBLEM.

231. *The length of a pendulum vibrating seconds in any latitude being given, to find the length of a like pendulum in any other latitude.*

We have shewn, in *Art.* 107, that the lengths of pendulums vibrating seconds in any latitude are as the force of gravity in that latitude, and in *Art.*219, that these forces are inversly as $\sqrt{1 - nrr}$, when r denotes the sine of the latitude; and therefore at the equator, unity is to $\sqrt{1 - nrr}$ as the length of a pendulums vibrating seconds, in the latitude whose sine is r, to the length of a like pendulum at the equator.

EXAMPLE.

232. Mr. De Mairan found by a great number of very accurate experiments, that the length of a pendulum vibrating seconds at Paris is 440.567 lines French measure: a line is the twelfth part of an inch,

and as $\sqrt{1 - nrr} = .99738$ in that latitude, by *Art.* 227, we have 1 : 99738 : : 440.567 : 439.4127 lines for the length of a pendulum vibrating ſeconds at the equator.

Sir Iſaac has found this length to be 439.468, as may be ſeen in his table, page 247, Eng. edit. but he obſerves afterwards, page 250, that the earth is more elevated at the equator than he had computed it; the length of his pendulum muſt of courſe be ſhorter: on the contrary, Bouguer makes it 439.21 lines only, page 342; but we ſhall ſhew hereafter for what reaſon he differs ſo much in his meaſure from other authors.

EXAMPLE.

233. *The length of a pendulum vibrating ſeconds at the equator being given, to find the length of a like pendulum at Pellow in the latitude of* 66° 48′.

We have $\sqrt{1 - nrr} = .996$ nearly, by *Art.* 227, and therefore .996 : 1 : : 439.4127 : 441.177 lines nearly, by *Art.* 231. This agrees with that found by Maupertuis, p. 226, Eng. edit.

EXAMPLE.

234. *The length of a pendulum vibrating ſeconds at the equator being given, to find the length of a like pendulum at London in the latitude of* 51° 32′.

We have $\sqrt{1 - nrr} = .99716$, by *Art.* 228, and hence .99716 : 1 : : 439.4127 : 440.6642 lines for the length required. Now as the French foot is to the Engliſh foot as 114 is to 107, this length will be 469.4927 lines, Engliſh meaſure; or dividing by 12, 39.1244 inches, which wants but .0006 parts of an inch of the length 39.125 found by Dr. Halley, from ſeveral accurate experiments.

The ſame length would have been found, if it had been deduced from that at Paris.

The

The near agreement between the lengths of pendulums deduced from theory, with thoſe found by experiments, ſhews how nicely the experiments muſt have been made, and how accurately we have determined the ratio between the axis and the diameter of the equator.

The length of a pendulum vibrating ſeconds at London, ſaid to have been found by Sir Iſaac, is 39.207 inches, which is abſolutely erroneous. For when converted into French meaſure gives 441.594 lines; whereas that at the pole is only 441.387, as determined by Sir Iſaac, as may be ſeen in his table, page 247.

EXAMPLE.

235. *The length of a pendulum vibrating ſeconds at the equator being given, to find that of a like pendulum at Edinburgh in the latitude of* 55° : 47′.

We have found $\sqrt{1 - nrr} = .9968$, in *Art.*229; and hence .9968 : 1 : : 439.4127 : 440.809 lines for the length required; which, when reduced into Engliſh meaſure, gives 39.137 inches.

EXAMPLE.

236. *The length of a pendulum vibrating ſeconds at the equator being given, to find that of a like pendulum at Dublin in the latitude of* 53° : 11′.

Here we have $\sqrt{1 - nrr} = .99703$, by *Art.*230; hence .99703 : 1 : : 439.4127 : 440.7216 lines for the length required; which, being reduced into Engliſh meaſure, gives 39.1294 inches.

EXAMPLE.

237. *The length of a pendulum vibrating ſeconds at the equator being given, to find that of a like pendulum at the pole.*

Here we have $\sqrt{1-nrr}=b=\frac{215}{216}$, and therefore $\frac{216}{215}\times 439.4127 = 441.4564$ lines for the length required; which exceeds that at the equator by 2.0437 lines.

PROBLEM.

338. *To find the length of the radius of the equator, and that of half the axis.*

The radius of curvature at the equator B, added to the value of CK, when the point M falls on the point B, gives the radius CB of the equator. Now since the meridian degree at the equator is 56753 toises, this, multiplied by 180, gives 10215540 toises for half the circumference; which, being to the radius as 355 to 113, gives 3251707 toises for this radius.

But when the point M falls on the point B, the radius of curvature MK is to the value of CK as bb to n, by *Art.* 219; and since $bb=\frac{46225}{46656}$, and $n=\frac{431}{46656}$, by *Art.* 226, we have 46225 : 431 :: 3251707 : 30319 for the value required; which, being added to the radius of curvature at the equator, gives 3282026 toises for the radius CB of the equator, and the semi-circumference 10310789 toises, and one degree 57282 toises; which therefore exceeds the meridian degree 56753 by 529 toises.

As CB is to CA in the ratio of 216 to 215, we get 3266830 toises for half the axis CA. An English mile is 1760 yards or 829 French toises, the radius of the equator will be 3959.01 English miles, half the axis 3940.68 miles, and their difference 18.33 miles.

Sir Isaac found this difference 17.1 miles, page 243; but observes, in page 250, that the earth is a little higher at the equator than what he had computed it. This shews that our difference agrees with Sir Isaac's; whereas Maclaurin, Simpson, and Clairaut, make it less, and Bouguer much greater.

PROBLEM.

PROBLEM.

239. *To find the force of gravity at the equator.*

We have found that a body falls through a ſpace of 15.097 in a ſecond at Paris, by *Art.* 107, and that the force of gravity at Paris is to the force of gravity at the equator as unity to $\sqrt{1 - nrr} = .99738$, *Art.* 227; hence 1 : .99738 : : 15.097 : 15.057446 feet, for the height fallen through at the equator in the ſame time: but if we ſuppoſe the ratio of the axis to be as 230 to 231, then will this height be 15.059967 feet.

PROBLEM.

240. *To find the centrifugal force at the equator.*

Let r denote the circumference of a circle whoſe diameter is unity, a the radius of the equator, and t the time of one revolution; then will $2ar$ expreſs the circumference of the equator; which, divided by the time t, gives $\frac{2ar}{t}$ for the arc deſcribed in a ſecond, when t is expreſſed in ſeconds. Now $r = 3.14159$; and as $t = 86164$ ſeconds, and $a = 3282025$ toiſes; we get $\frac{2ar}{t} = 239.3293$ toiſes for the arc deſcribed in a ſecond, and the ſquare 57278.513849 of that arc, divided by the diameter 6564050, and multiplied by 6 feet, gives .052356 feet for the verſed ſine of that arc, or the ſpace deſcribed by the centrifugal force in a ſecond. And ſince the force of gravity deſcribes a ſpace of 15.059967 feet, by the laſt *Art.* at the ſame time, the centrifugal force is to the force of gravity as .052356 to 15.059967, or as unity to 287.597.

If we ſuppoſe the ratio of 230 to 231 of the axis, then the radius of the equator will be 3280044 toiſes; and by the ſame proceſs as before, the gravity at

at the equator will be to the centrifugal force at that place as 287.81 to unity.

And if we admit Bouguer's ratio of 178 to 179, the gravity at the equator will be 15.04914 feet, the radius of the equator 3288345 toiſes, and the gravity at the equator to the centrifugal force as 286.885 to unity.

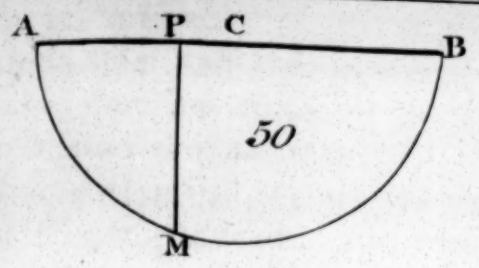
Obſervations on the authors who have wrote on the figure of the earth.

Maclaurin, Thomas Simpſon, and Clairaut have found the ratio of the axis to the diameter of the equator to be as 230 to 231, which is leſs than that of Sir Iſaac Newton, without conſidering whether theirs agreed better with the ſeveral meaſures of a meridian degree, or the moſt accurate obſervations on the length of pendulums, which yet is neceſſary to aſcertain the truth of all ſuch kind of ſpeculations; it is true, they have made uſe of the moſt ſublime part of the mathematics in order to ſhew their great ſkill and knowlege in computations, without conſidering that knowledge deduced from the moſt ſimple principles is leaſt liable to miſtakes, and more ſatisfactory; though they beſtowed ſo much pains in their theory and algebraic calculations, they are in my opinion not ſufficiently exact in their numeral approximations: for Simpſon, in page 472, finds the ratio of the axis to the diameter to be as 1151 to 1156, which he ſays is as 230 to 231; whereas, by the common rule of approximation, it is leſs than that of 231 to 232, and greater than that of 232 to 233, but neareſt to this laſt.

Clairaut, page 192, finds this ratio to be as 2304 to 2314, which he makes as 230 to 231; whereas it is greater than this, and leſs than that of 231 to 232; but neareſt of all as 461 to 463: this author, finding the above ratio not to be exactly that of 230

to

Plate V



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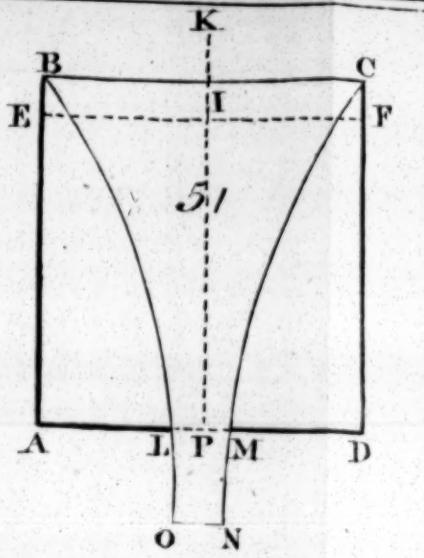

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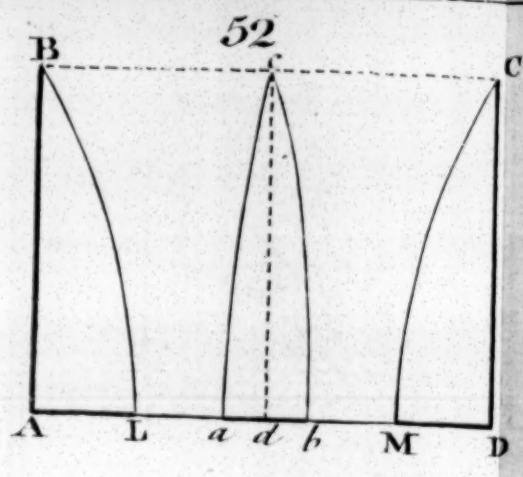

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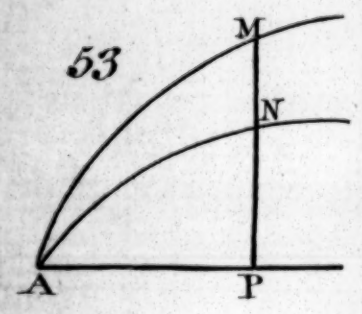

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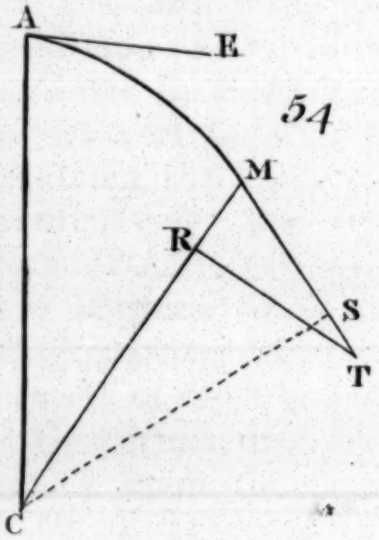

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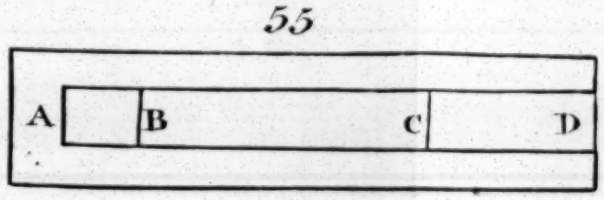

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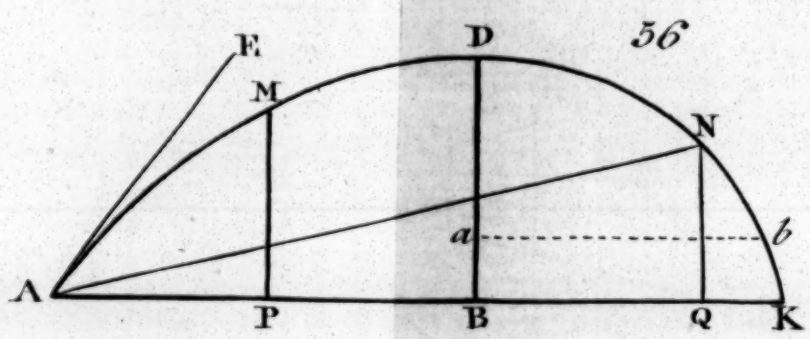

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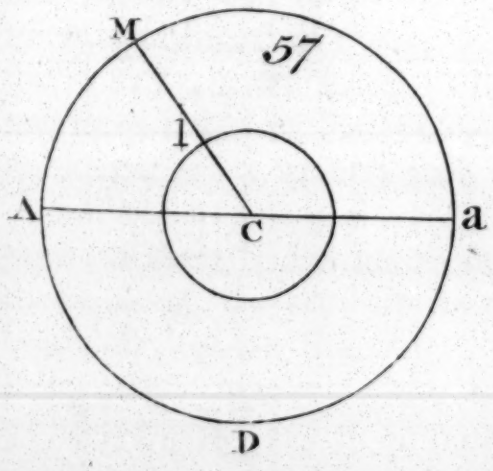

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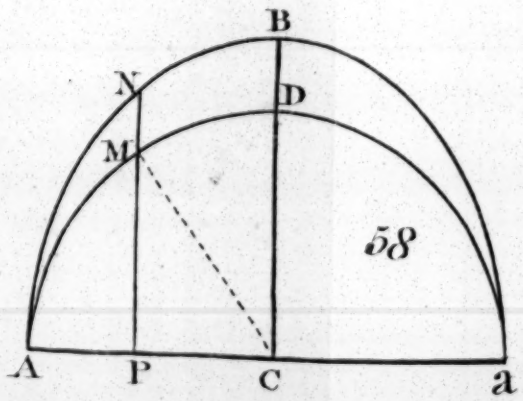

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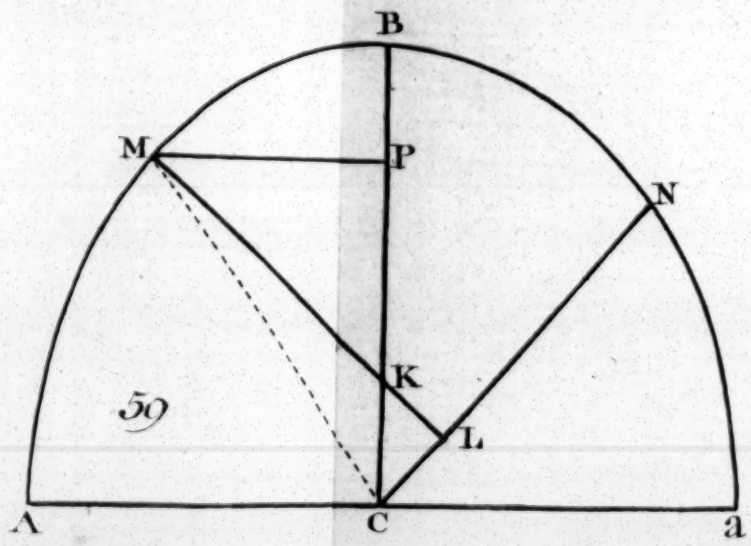

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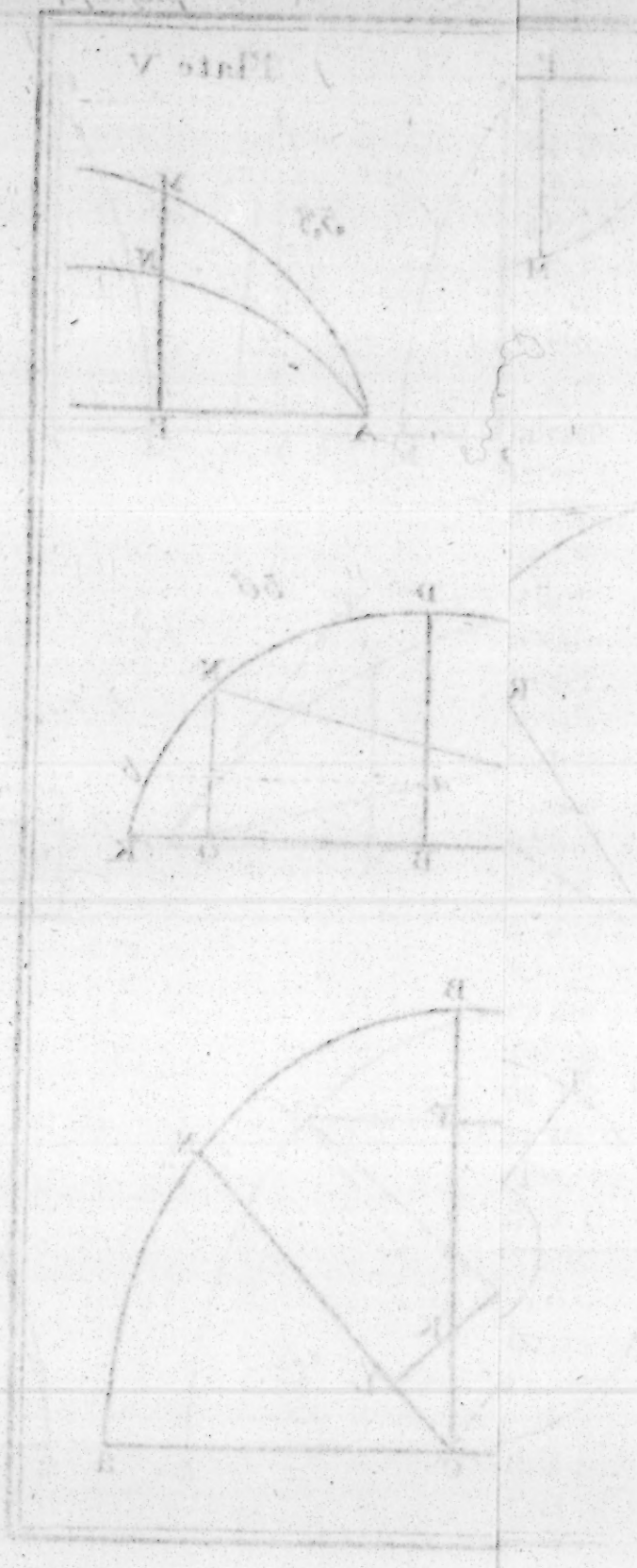

to 231, by ſuppoſing the ratio of the centrifugal force, and that of gravity at the equator, and to find this laſt he aſſumes the ratio of the axis, whereby he finds it to be as unity to 287.5; granting that the ratio of the axis is right, and his calculation juſt, then this laſt ratio is true; but it does not prove that the ratio of the axis is true by any means: had he found the ratio of the centrifugal force to that of gravity, without ſuppoſing that of the axis, the laſt muſt have been true alſo: a pleaſant way of arguing backwards and forwards from one unknown thing to another, and at laſt draw a concluſion to have arrived at truth. This author has no better ſucceeded in page 188, where he ſays that the ratio of the axis of 230 to 231 is true, upon ſuppoſing the earth to be uniformly denſe, and in no other caſe; and if it is not ſo, that ratio muſt be leſs and could not be greater: whereas we have found it to be greater, that is, as 215 to 216, and ſhewn it to agree better with the meaſures and obſervations than any of thoſe hitherto given by others.

Maupertuis, in page 162, Engliſh edit. makes a remark upon the degree meaſured by Picard in France, and reduces it from 57060 toiſes to 56925.7, the difference is no leſs than 134.3 toiſes; which is ſtill leſs by 366 toiſes than that of Picard's 57292 toiſes, publiſhed in 1701, and leſs by 257 toiſes than that 57183, corrected. This diſagreement of the meaſures made in France by the moſt conſiderable mathematicians, creates ſuch a confuſion as renders it impoſſible to judge the right from the wrong; had they not been more ſucceſsful in the north and near the equator, the true figure of the earth would have been leſs known than ever: what is ſurpriſing, Sir Iſaac came nearer to the truth than all together, without having any good meaſure to depend upon.

We

We now come to Bouguer's work, the laſt publiſhed, which one ſhould think would decide the queſtion beyond all exception; but he makes the equator higher than it is, whereas the others make it too low. In page 296, he ſays, that by taking the meaſures at the equator and near the pole for granted, he found the ratio of the axis to be as 215 to 214; but was obliged to change his opinion, when he heard that the meridian degree near Paris had been corrected by the gentlemen who had been employed near the pole, and made it 57183 toiſes, whereby he found the ratio of 223 to 222, in which he was confirmed by the meaſure of a degree of longitude in the latitude of 43° 32′; which was found to be 51618 toiſes by meaſure, and 41607 by computation.

In page 299, he changes again when he heard that Picard's meaſure, inſtead of being 57183 toiſes, had been corrected again to 57074; from whence, in page 304, he concludes that the true ratio of the axis was that of 178 to 179. Now as his firſt ratio, deduced from the ſame datas as we did, came out nearly the ſame as ours, becauſe we found that the true one was between that of 214 to 215, and of 215 to 216; but nearer the laſt: had he remained ſatisfied, he would have been right; but the various meaſures of a meridian degree in France, on which he depended too much, was what miſled him.

To ſhew that his ratio is not true; the difference between the ſquares of 179 and 178 gives $n = \frac{357}{32041}$, by *Art.* 224, the ſquare of the ſine of 66° 20′, is $rr =$ .83885, by the ſame *Art.* and hence $\sqrt{1 - nrr} =$ .99532, whoſe cube is .98602; whence .98602 : 1 : : 56753 : 57557.6 toiſes for the meredian degree near the pole; which exceeds the meaſure 57438 by 119.6 toiſes, and if his own correction of that degree be allowed the difference will be 1361 toiſes.

Again, in the latitude of Paris, we have $rr = .56699$, by *Art.* 227, and hence $\sqrt{1 - nrr} = .99683$, whoſe cube is 99032; therefore .99052 : 1 : : 56753 : 57296 toiſes for the degree at Paris, which exceeds the meaſure corrected 57074 by 222 toiſes.

This evidently proves his ratio of the axis is the fartheſt of all from the true one: it does not appear from his work, that he ſo much as thought of comparing his ratio with any of the meaſures, no more than the obſervations with pendulums; for the length of a pendulum vibrating ſeconds at the equator is, according to his ratio, 439.17 lines; which not only diſagrees with all others, but his own obſervations.

With regard to the experiments, he ſays, the weight of his pendulum was one ounce five drams and two ſcruples, ſuſpended by a fibre of an aloe leaf, and faſtened above by a pincher ſcrewed into a wall; the form of the weight was two equal fruſtums of a cone joined by their greateſt baſes, whoſe center of oſcillation, he ſays, was rather above the middle, becauſe of the rod's weight, 10 toiſes of which weighed 4 grains; ſo that 3 feet weighed $\frac{1}{5}$ of a grain.

Now, whether ſo ſmall a weight cauſes the center of oſcillation of the bob to riſe from below the center above it, is not very probable; the rod of the pendulum being fixed above cannot move ſo freely as about the center, let it be ever ſo ſupple; beſides a ſmall weight is more reſiſted in proportion in its motion than a heavy one. As his length is leſs by .2 of a line, or $\frac{1}{60}$ part of an inch, than it ſhould be, which can be owing to nothing elſe than ſo imperfect an experiment; beſides he judged by the eye of the length, and a ruler placed by the rod. But rather than allow of the imperfection of his experiment, he ſuppoſes that the univerſal law of gravity does not hold good under the equator.

THE END.

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ARTILLERY DICTIONARY.

AFFUT, the French name of a gun carriage, for which we have no proper name; the only diſtinction from all other carriages, is that it belongs to a gun.

AMMUNITION, comprehends powder, ſhot, ſhells, carcaſſes, and every neceſſary in the field for the uſe of fire-arms.

AMMUNITION WAGGON, is a four-wheel carriage with ſhafts, and having the ſides railed in with raves and ſtaves, lined with wicker work, ſo as to carry bread and all ſorts of tools. See plate xx.

As the fore horſes draw two a-breaſt, that in the ſhaft cannot go in the ſame tract, it continually treads on one ſide or other the ridge left between the two; or if the road is quartered it ſhuns to tread in the rut, by which the wheels can never follow the ſame tract; for which reaſon it would be more convenient to have two pair of ſhafts, as waggons that carry great loads.

AMMUNITION CART, is a two-wheel carriage with ſhafts, and the ſides as well as the fore and hind parts are incloſed with boards inſtead of wicker work.

APRON, a ſmall thin piece of lead about ſix inches ſquare, which is laid over the vent of a gun or mortar when loaded, till they are fired, to prevent the priming from dampneſs.

ARSENAL, a large building that contains ſtore-rooms for every thing uſed in artillery.

ARMS, the two round ends of an axle-tree. See axle-tree.

ARTILLERY, comprehends cannon, mortars, howitzes, petards, ſhot, ſhells, carcaſſes, and all their appurtenances uſed at home and in the field.

ARTILLERY, REGIMENT, is compoſed of three battalions (and a company of cadets, of which the maſter-general is always captain) each of them commanded by a colonel, lieutenant-colonel, and a major; the maſter-general of the ordnance is commander in chief; the lieutenant-general, commander en ſecond; and the three

colonels are called colonel-commandant each of his battalion; each company contains matroſſes, gunners, and bombardiers.

ARTILLERY, EQUIPAGE, is a quantity of guns, mortars, ſhot, and ſhells, with all the neceſſary ſtores made out for a campaign, or an expedition by ſea or land.

ASTRAGAL, a moulding, in architecture, like a round ring with a fillet on each ſide, uſed in guns and mortars. See guns.

AXES are of two ſorts, one for felling wood, and a pick-axe for picking ſtones or digging trenches when the ground is too hard for the ſpade.

AXIS, in mechanics, is that line or beam about which any thing turns.

AXLE-TREE of a carriage about which the wheels turn, the middle part between the wheels is called the body; the two ends, which are round and enter into the naves, the arms.

They are made of dry young elm, beach, or oak.

AXLE-TREE BAR, in travelling carriages, is a flat iron bar that reaches from one end to the other, and let into the wood underneath quite fluſh.

AXLE-TREE BOLTS is an iron bolt paſſing through the middle of the axle-tree to bolt the axle-tree bar.

AXLE-TREE BANDS, uſed in field carriages, they go round the ends of the body, and are bolted to the underſide of the ſide-pieces of the carriage.

AXLE-TREE HOOPS, iron rings that go round the ends of the arms to faſten the iron bar, when there are any, and to ſecure the linch-pin holes.

AXLE-TREE STAYS, are uſed in ſhip or garriſon carriages, made of iron in the form of an *S* to faſten the fore axle-tree firm to the underſide of the ſide-pieces, to prevent their giving way in the ſudden jerk occaſioned by the recoil when the gun is fired.

BAGGAGE WAGGONS are common waggons railed in with raves and ſtaves, and have bales to cover the top, with ſtrong canvaſs to keep the baggage from wet.

BAGS, SAND, they are 16 inches diameter and 30 high, filled with earth or ſand to repair breaches and the embraſures of batteries, when damaged by the enemies fire, or by the blaſt of the guns.

BAGS are alſo uſed on many occaſions to hold ſmall ſhot, flints, &c.

BALLS, CANNON, formerly made of ſtone but now of iron, they are fired out of cannons againſt an enemy, or againſt their works made for cover.

BALLS, FIRE, made of an iron ſhell, or paper, filled with a ſlow compoſition, to throw into the enemies works at night to ſee what they are doing, and to enable the gunners to point their guns on them.

BANNERS, or the ordnance flag fixed upon the fore part of the drum-major's kettle-drum carriage.

BARRACKS, a building to lodge ſoldiers in.

BARRELS, POWDER, are about 16 inches diameter, and 30 or 32 long, and hold 100 pounds.

BARREL, BUDGE, holds from 40 to 60 pounds; at one end is fixed a leather bag with braſs nails: they are uſed upon batteries to keep the powder from firing by accident, for loading the guns and mortars.

BARROWS, to carry loads from one place to another.

BARROWS, HAND, ſerve for carrying loads by two men.

BATTERY, to place guns or mortars on in a ſiege; it conſiſts of a breaſt-work or parapet of 20 or 22 feet thick, and 7 or 8 feet high, with embraſures to fire the guns through, behind which is a platform of wood 18 feet long, 8 feet broad before, and about 16 behind, ſloping about ſix inches forwards in order to prevent the guns from recoiling too much, and bringing the guns eaſier forward when loaded.

They commonly allow 18 feet for the diſtance from gun to gun, and when the battery is ſeen on the ſides by the enemies works they add flanks of 10 or 12 feet thick, from 15 to 18 long, and as high as the parapet. They are called by different names from the ſeveral manners of firing.

BATTERY, BARBET, when the breaſt-work is only three feet high, ſo that the guns may be traverſed any way, as are commonly thoſe near the ſea or large rivers, to fire at the ſhips as they are coming, paſs by, or paſſed; theſe batteries are alſo uſed upon the faces of the baſtions in a ſiege while the beſiegers are as yet at a diſtance, and becauſe embraſures would help to make the breach in the faces ſooner than without them.

BATTERY, RICOCHET, which are placed about 100 yards before the firſt parallel, perpendicular to the faces of the works, to enfilade them by giving the guns a ſmall elevation to go over the oppoſite parapet, and loaded with a ſmall quantity of powder.

This method of firing was invented by M. Vauban, and firſt put in practice at the ſiege of Aeth in 1697.

BATTERY MORTAR, they have a breaſt-work as before, without embraſures; the inſide ſlope of it ought to be ſuch that two pickets might be placed, one above and one in the middle, to mark the direction of the object. The inſide of theſe batteries are ſunk two or three feet into the ground, by which they are much ſooner made than thoſe of cannon.

BATTERING PIECES ſerve to make breaches; it is judged by moſt nations that no leſs than a 24 pounder is proper for that uſe; formerly much larger calibers were uſed, but as they were ſo long and heavy, became very troubleſome to carry and manage, they have been rejected; but if larger calibers can be made ſo as to be worked and managed with eaſe, there is no doubt but that they will make a breach ſooner than a 24 pounder; this we have proved to be very poſſible, and what is more of iron inſtead of braſs, as they have hitherto been made.

BAYONET, a broad dagger without any guard, generally made with a round hollow handle, to fix to the muzzle of a muſket, in which manner it ſerves inſtead of a pike to receive the charge of horſe or foot.

BEDS, MORTAR, they are made of ſolid timber, conſiſting generally of four pieces faſtened together with ſtrong iron bolts and bars; thoſe of the royal and coehorns are made of one ſolid block only.

BEDS, SEA MORTAR, theſe are likewiſe made of ſolid wood like the former, but differ in their form, having in the middle a hole to receive the pintle or ſtrong iron bolt about which they turn.

BED, STOOL, is a piece of wood to lay the breech of a gun upon in a truck-carriage, with another piece fixed to it at the hind end, that reſts upon the body of the hind axle-tree, and the fore part is ſupported by an iron bolt.

BOLSTER, is that piece of wood faſtened upon the axle-tree of any wheel carriage.

BOLT, made of iron, to bolt wood or iron together; they receive additional names from the different parts of a carriage where they are uſed.

BOLT, AXLE-TREE, thoſe which pin the hind axle-tree of a truck-carriage to the ſide-pieces.

BOLT, EYE, the fore bolt of a gun-carriage, or of a mortar-bed, to lock the cap-ſquare by means of a key.

BOLT, JOINT, is that to which the cap-ſquare is fixed by means of a hinge.

BOLT, TRANSOM, is that which goes through the tranſom of a gun-carriage to faſten the ſide pieces firmly to the tranſoms.

BOLT, BED, that which paſſes through the ſide of a truck-carriage, and ſupports the fore-end of the ſtool bed.

BOLT, BREECHING, with rings, uſed in ſhip-carriages to faſten the tackles which ſerve to ſtop the recoil of the gun, and to advance it when loaded.

BOLT, BOLSTER, that which faſtens the bolſter of a limber, or of any waggon to the bed under it, or to the axle-tree.

BOLT, BRACKET, thoſe which faſten the upper and under parts together of the cheeks of a truck-carriage.

Bolt, Stool Bed, there are two in each to faſten the ſtool to the bed, riveted underneath.

Bolt, Garnish, thoſe which are uſed in field-carriages, whoſe heads terminate in a point, and have a neck under it, and faſten the fore part of the axle-tree band by means of a key.

Bombardier, the third rank of private men in a company of artillery.

Bomb-Ketch, a veſſel that carries ſea mortars, each two, a 10 and a 13 inch one.

Bomb-Tender, that veſſel which carries the neceſſary ſtores for the uſe of the mortars.

To bombard a place, is to fling a great number of ſhells in it, either to deſtroy the buildings or to oblige it to ſurrender.

Bore of a gun or mortar, is the hollow part which receives the powder and ſhot in the guns, or the ſhell in the mortar.

Box, they are of ſeveral ſorts, and for different uſes.

Boxes, Nave, made of iron, and faſtened one at each end of the nave, to prevent the arms of the axle-tree about which the boxes move, from cauſing too much friction.

Boxes, Wood, with lids, which ſerve to hold grape ſhot; each caliber has its own, diſtinguiſhed by marks of the caliber on the lid.

Boxes, Tin, to fill with ſmall ſhot for grape, the ſize of the gun.

Boxes, Wood, made to ſlip on and off over the axle-tree of light field carriages, to hold 12 cartridges for firing upon a march in caſe of an attack.

Breast, the fore ends of the ſide-pieces of a travelling-carriage; thus, breaſt tranſom, breaſt tranſom plates with hooks.

Breech of a gun, is that ſolid piece of metal behind, between the end of the bore and the ſection, through the end of the baſe ring, or which terminates the hind end of the gun.

BURR, a round iron ring which ſerves to rivet the end of the bolt ſo as to form a round head like that of the bolt.

BUTTON of the caſcable, in a gun or howitz, is the hind part of the piece made round in the form of a ball.

BULLET, of lead for muſkets, carabines, piſtols, and grape ſhot.

CALIBER of a gun or mortar, is the ſame as their bore or opening; and the diameter of the bore is called the diameter of its caliber.

CAMP, is the ſpot of ground where a ſtrong party of troops, or the army, ſettle and pitch their tents. The artillery camp is always upon the higheſt ſpot before or behind that of the army, to have room for placing their artillery and ſtores.

CANNON. See guns.

CANTEENS, a tin veſſel uſed by the ſoldiers to carry their drink in.

CAPS, made of leather for covering the mouth of a mortar when loaded till it is fired, to prevent damps getting into it.

CARRIAGE, a general name for all machines that have wheels.

CARRIAGE, GUNS, the wooden frame upon which guns are fixed; there are three ſorts commonly uſed, viz.

CARRIAGE, FIELD, or travelling, an aſſemblage of two long planks called ſide-pieces, joined together by three or four pieces called tranſoms, one end is ſupported by two wheels and the other touches the ground.

CARRIAGE, SEA, made of two ſide-pieces, one tranſom and bolts, and has four ſmall wheels, each made of one piece of wood called trucks.

CARRIAGE, GARRISON, made the ſame as the former, with this difference, that their wheels or trucks are made of caſt iron.

There has been ſome of theſe carriages made entirely of caſt iron, and are eſteemed very good, eſpecially

cially in hot climates, where the wood is ſaid to rot very ſoon.

CARTRIDGE, a bag to hold as much powder as is ſufficient for loading a gun, they are made of a ſtrong paper called by that name, or of parchment; but as the end generally remains in the bottom of the gun, the paper ones retain fire, and may fire the next, which makes it dangerous; and that made of parchment ſticks cloſe to the bottom and accumulates, if not removed, till it ſtops the vent, which it fills and ſticks ſo faſt in it as not to be removed without an iron drill; to prevent thoſe inconveniencies, to the parchment caſe is fixed a flannel bottom, by which all accidents are avoided; but as the duſt of powder may paſs through the flannel and thereby cauſe accidents, a parchment cap is put over it to take off before the cartridge is put into the gun.

CARTRIDGE, for muſkets, carabines, and piſtols, are made of thin paper, ſo as to hold half an ounce or leſs of powder and a bullet.

CARTS, are two-wheel carriages to carry powder, tools, and all kind of ammunition.

CART, SLING, ſerves to ſling a gun under it, and to carry it at a ſmall diſtance.

CART, POWDER, incloſed round with boards, with lids on the top like a roof, and covered over with an oil-cloth to prevent any dampneſs coming to the powder.

CART, FORGE, ſerves to carry an anvil, tongs, hammers, and a poker upon a march, for mending any iron work that may break upon a march, or repair them in the field.

CART or TUMBREL, ſerves to carry all ſorts of tools and other ſmall neceſſaries to the artillery.

CENTER, the middle part of the ſide-pieces in a traveling carriage at the bending part.

CIMAISE, a moulding of architecture, applied in our conſtruction of guns, near the mouth of the gun.

COLLAR,

COLLAR, the part of tackle of a horſe that goes about the neck, or a belt that goes over the ſhoulder of a man, and the arm on the other ſide, when they draw a gun for a little way.

COINS, or WEDGES, to lay under the breech of a gun to raiſe or depreſs it.

COLOURS, uſed in the camp, with ſtaves, to mark the front line of the parade, and in marking out the camp.

COLOURS, or ENSIGNS, of the various regiments to diſtinguiſh them.

The artillery colours are three guns upon their carriages, which are alſo called the ordnance arms.

CROSSS-BARS, in a limber or waggon, the two wooden bars mortiſed into the hind part of the ſhafts.

CROWS, are iron bars, a little bent on one end, and made uſe of as hand-ſpikes.

CROWS FEET, an iron of four points of about ſix inches long, which are uſed againſt the cavalry, for one point will always be uppermoſt, let it fall as it will.

DIMENSIONS, the length, breadth, and thickneſs of bodies.

DOG-NAILS, have a round flat head, and is itſelf round in part near the head.

DOLPHINS, are the two handles placed on the ſecond reinforce ring of braſs guns, reſembling the fiſh of that name.

DOWLEDGES, iron plates let into the fellows of a wheel on the inſide, and faſtened with four iron pins, rivetted on the other ſide.

DOWL BARS, are two iron beams, placed in mortar beds in a vertical poſition to faſten the upper and under beds together.

DOWL PINS, are made of wood, about three inches long, one in diameter, placed in the middle of the joints of the fellows of a wheel to keep them toge-

ther. Theſe pins are alſo uſed in ſhip carriages, two on each ſide, to keep the ſide-pieces together.

DRAUGHT HOOKS, fixed on the ſides of travelling carriages to the tranſom plates, three on each ſide: ſometimes a draught hook is alſo joined to the waſher.

DRAUGHT RINGS, fixed to the ſide-ſtraps of a travelling carriage by means of ſtaples rivetted on the inſide.

DRIVER, a wooden cylinder for driving of fuſees.

DRUM, a military inſtrument of muſic, to beat to arms, or to the ſeveral military motions.

DRUM, KETTLE, uſed by the cavalry as likewiſe in the artillery.

KETTLE DRUM CARR, of artillery, is a four wheel carriage drawn by four horſes; on the fore-part ſtands the ordnance flag, and on the hind ſits a drum-major with two kettle drums as in a chair of ſtate.

EMBRASURE, that opening left in the epaulement for the guns to fire through.

ENGINE, to drive fuzes, conſiſts of a wheel with a handle to it, to raiſe a certain weight, and to let it fall upon the driver, by which the ſtrokes become more equal.

ENGINE, to draw fuzes out of ſhells, has a ſcrew fixed upon a three legged ſtand; the bottom of this ſtand has a ring to place it on the ſhell, and at the end of the ſcrew is fixed a hand ſcrew by means of a collar, which being ſcrewed on the fuze by turning the upper ſcrew draws or raiſes the fuze.

ENGINE, for driving piles by means of horſes, an ingenious invention of an acquaintance of mine for driving the piles at Weſtminſter bridge.

ENGINE, for raiſing of water, either by fire, as at York Buildings or near Chelſea, others by means of water wheels.

Esses, Fixed, to draught chains made in the form of an S, one end is faſtened to the chain, and the other to hook to the horſes harneſs or to a ſtaple.

Eye Bolts, ſee Bolt.

Fathom, a meaſure called ſo, being equal to two yards or ſix feet, equivalent to the French word toiſe.

Fellows, the parts of a wheel which forms its circumference, made of dry elm; there are generally ſix in each wheel.

Flags, with ſtaves, the union and the red, moſtly uſed on board ſhips.

Flambeaux, uſed in the artillery, at night, upon a march, or in the park.

Flints, for muſkets, carabine, and piſtols.

Fore-Lock Key, the flat iron, broader at one end than at the other, which paſſes through the eye-bolt to faſten the cap-ſquare, and to ſecure the trunnions of a gun or mortar.

Formers, of different ſizes, a cylinder of wood, uſed in laboratory work, to drive the compoſitions of fuzes and rockets.

Foundery, for caſting guns, mortars, ſhot, and ſhells.

Funnels, of different ſizes, to pour the powder into ſhells, and the compoſition into fuze or rocket caſes.

Galloper, a carriage of a light field-piece with ſhafts, to be drawn without a limber, but very little uſed now.

Garnish Bolt, or nails. See bolts and nails.

Garrison, the troops quartered in a fortified place to guard and defend it in caſe of an attack, are called collectively by that name.

Gin, an engine to raiſe guns compoſed of three long legs; two of them are kept at a proper diſtance by means of two iron bars fixed on one of the legs by a ſtaple paſſing through the hole at one end; the other end has a hook which enters into a ſtaple fixed into the other leg ſo as to put off or on at pleaſure; at

three feet from the bottom is a roller upon which the cable is wound, and the three legs are joined together with an iron bolt, about which they move; to this bolt is alſo fixed an iron half ring to hook on a windleſs; when the gin ſtands upright ſo as the legs ſtand at a proper diſtance, one end of the cable is faſtened to the gun, and the other paſſing through the pulleys and round the roller, the roller is turned round by means of hand-ſpikes paſſing through the holes in the ends of the roller; whilſt a man holds the cable tight, the gun is raiſed to the height that the carriage may be put under it.

GRAPE-SHOT, is a certain number of ſmall iron ſhot in a bag of coarſe cloth, juſt to hold a bottom of iron or wood with a pin in the middle; about this pin are put as many ſhot as the grape is to conſiſt of; then they are quilted with a ſtrong packthread in order to keep the ſhot tight; their ſize is according to the bore of the gun, and their weight nearly equal to that of the ſhot of that gun.

Small iron or lead, the ſize of muſket ſhot, are alſo put into tin boxes, to ſerve in the ſame manner as grape-ſhot.

GRAPLING IRONS, is compoſed of four, five, and ſix branches, bent round and pointed, with a ring at the root, to which is faſtened a rope to hold by, when the grapple is thrown at any thing, in order to bring it near ſo as to lay hold of it.

GUIDEFORE, in a waggon, are two crooked pieces of wood fixed upon the axle-tree each with an iron bolt, joined before by a croſs bar, to which the ſhafts are faſtened with a bolt and key, and upon the hind part is faſtened the ſweep bar.

GUIDEHIND, is likewiſe compoſed of two crooked pieces of wood faſtened to the hind axle-tree as the former, and faſtened to the pole with a flat ring and an iron bolt.

GUN, the length is diſtinguiſhed by three parts; the firſt reinforce, the ſecond reinforce, and the chace; the firſt

reinforce is two-ſevenths, and the ſecond one-ſeventh and half a diameter of the ſhot.

The inſide hollow wherein the powder and ſhot are lodged, the bore and the diameter of the bore is called the diameter of the caliber.

The part between the hind end and the bore, the breech; and the fore-part of the bore, the mouth.

The caſcable is the part terminated by the hind part of the breech, and the extremity of the button.

The trunnions, the cylindric parts of metal which project on both ſides of the gun, which reſt in the grooves, made in the ſide-pieces of a carriage.

The mouldings are thoſe behind the breech, and are looked upon to belong to the caſcable, the firſt and ſecond reinforce rings, ogees, aſtragals, and fillets.

Thoſe of the firſt reinforce are a ring ogee joining to it, and an aſtragal with fillets; the part of the gun between the ogee and aſtragal is called the vent-field, becauſe the vent is placed there.

The ogee of the ſecond, a ring and ogee; and thoſe of the chace, a ring-ogee; the aſtragal with fillets, the muzzle aſtragal, the ſwelling of the muzzle an ogee, or cimaiſe and two fillets.

The part between the ogee and chace aſtragal, the chace girdle; and the part from the muzzle aſtragal and the mouth, the muzzle.

Formerly guns were diſtinguiſhed by the names of ſakers, culverins, cannon, demi-cannon, &c. but at preſent their names are taken from the weight of their ſhot; as for example, a 12 or 24 pounder carries a ball of 12 or 24 pounds weight.

Guns are made of braſs or caſt iron; the braſs is a mixture of copper and tin: ſometimes yellow braſs is added, but is reckoned to make the metal brittle. The moſt common proportion is to an hundred pounds of copper twelve pounds of tin. But as copper requires a red heat to melt, and tin does melt in a common fire, when a gun is much heated by firing, the tin melts or

ſoftens

ſoftens ſo much that the copper alone ſupports the force of exploſion, whereby they generally bend at the muzzle, and the vent widens ſo much as to render the gun uſeleſs. If ſuch a compoſition of metal could be found as required an equal degree of heat to melt, it would anſwer the intent: but as no ſuch thing has been hitherto found, I look upon good iron to make better and more durable guns than any other compoſition whatever, as experiments and practice have ſhewn. For all our braſs battering guns made uſe of this laſt war were ſoon rendered unſerviceable, and iron ones ſubſtituted.

The neceſſary tools for loading and firing guns, are rammers, ſponges, ladles, worms, hand-ſpikes, wedges, or ſcrews.

The rammer is a cylinder of wood, whoſe diameter and axis is equal to that of the ſhot, and ſerves to ram home the wadds put upon the powder and ſhot; the ſponge is the ſame, only covered with lamb-ſkin, and ſerves to clean the gun when fired: the rammer and ſponge are fixed to the ſame handle.

The ladle ſerves to load the gun with looſe powder.

The worm ſerves to draw out the wads when a gun is to be unloaded.

The hand-ſpikes ſerve to move and to lay the gun.

The coins or wedges, to lay under the breech of the gun, and to raiſe or depreſs it.

In field-pieces, a ſcrew is uſed inſtead of coins, by which the gun is kept to the ſame elevation.

The tools neceſſary to prove guns, beſides thoſe mentioned for loading them, a priming iron, a ſearcher with a reliever, and a ſearcher with one point.

The firſt ſearcher is an iron, hollow at one end to receive a wooden handle, and on the other has from four to eight flat ſprings of about ſix inches long, pointed and turned outwards at the ends: the reliever is an iron flat ring, with a wooden handle, at right angles to it: when a gun is to be ſearched after it has been fired, this ſearcher is introduced and turned every way from one

end

end to the other; and if there is any hole, the point of one or the other ſpring gets into it, and remains till the reliever, paſſing round the handle of the ſearcher, preſſes the ſprings together and relieves it; and if any of the points catches in the vent, the priming iron is introduced to relieve it.

When there is any hole or roughneſs in the gun, the diſtance from the mouth is mark'd on the outſide with chalk.

The other ſearcher has alſo a wooden handle and a point at the fore end of about an inch long, at right angles to the length about this point is put ſome wax mixed with tallow, and when introduced into the hole or cavity, is preſſed in and drawn forwards and backwards, then the impreſſion upon the wax gives the depth, and the length is known by the motion of the ſearcher: if the hole is a quarter of an inch deep, and downwards, the gun is rejected.

GUNNER, is the ſecond rank of private men in the regiment of artillery.

HAIR-CLOTH, uſed on the floor of powder magazines, to prevent accidents of fire from mens ſhoes treading or rubbing upon or againſt ſand or gravel, &c. may ſtrike fire.

HAND-GRENADES, a ſmall iron ſhell from 2.5 to three inches diameter, filled with powder, which being lighted by means of a fuze, are thrown by the grenadiers amongſt the enemy when they come near.

HAND SCREWS, are compoſed of a toothed iron bar, that has a claw at the lower end and a fork at the upper; this bar is fixed in a ſtock of wood about 2.5 feet high, and moved by a rack-work; this claw or fork being placed under a weight raiſes it as far as the bar can go.

HAND-SPIKE, a wooden leaver five or ſix feet long, flattened at the lower end, and tapering towards the other, uſeful in moving guns to their place, after being fired and loaded again.

HAND-

HAND-MALLETS, a wooden hammer with a handle, to drive fuzes and picketts, in making faſcine batteries.

HAMMER, IRON, made of various ſizes and forms, the common have a head on one end and a claw on the other, to drive nails in or to draw them out.

HASP, a flat ſtaple to catch the bolt of a lock.

HATCHET, for cutting wood to make faſcines and pickets.

HEAD, the fore part of the ſide pieces of a gun carriage.

HELVE, or HAFT, the wooden hands of hatchets, hammers, or pick-axes, &c.

HINGES, are two iron bands, with a joint, nailed to the doors or lockers to faſten them, and move them back and forwards.

HOOKS, thoſe which are fixed to the tranſom plates of a field-carriage; they ſerve to fix chains for drawing it occaſionally back and forwards.

HOOPS, IRON, of ſeveral ſorts, ſuch as nave and axletree hoops.

HORN-BEAM, a wood much uſed for making the fuzes of ſhells.

HORNS, POWDER; the gunners ſling them over the ſhoulders with a belt, to prime the guns or mortars.

HOWITZ, a ſort of mortar, mounted upon a field-carriage like a gun; the difference between a mortar and a howitz is, that the trunnions of the firſt are at the end, and in the middle in the laſt.

HURTERS, a flatted iron fixed againſt the body of an axletree with ſtraps, to take off the friction of the naves of wheels againſt the body.

HURTER, of wood, is a piece of five by ſix inches ſquare, and eight feet long, laid upon a platform againſt the breaſt work, to ſtop the wheels from damaging the faſcines.

JOINT-BOLTS, by which one end of a cap ſquare is fixed to the carriage ſea bolts.

IRON-GUNS, firſt made of hammered, but now of caſt-iron.

IRON, PRIMING, which the gunners carry with their horns to clear the vent, or to releaſe the ſearcher in the proving of guns.

IRON, for marking with a broad R the horſes that are taken for the government's ſervice to draw the artillery ammuntion and ſtores.

KETTLE or COPPER, to boil compoſitions for fire-works.

KEYS, FORE-LOCK, ſerve to paſs through the lower end of bolts, to faſten them.

KEYS, with chains and ſtaples, fixed on the ſide-pieces of a carriage or mortar beds, they ſerve to faſten the capſquares, by paſſing through the eyes of the eye-bolts.

KEYS, SPRING, they ſerve the ſame purpoſes as the former, but inſtead of being one ſingle piece they are of two, like two ſprings laid one over another; when they are put into the eye-bolts they are pinched together at the ends, and when they are in, open again, and cannot ſhake out by the motion of the carriages; they are uſed in travelling carriages.

LABORATORY is the place where the different materials for fire-works are lodged and prepared. In the field a tent is uſed for that purpoſe pitched on a place where the enemies ſhot or ſhells cannot approach.

LADLES, made of copper to hold the powder for loading of guns, with long handles of wood, when cartridges are not uſed.

LADLES, SMALL, of copper; with ſhort handles, they fill the fuſes of ſhells, or any other compoſition to fill the caſes of ſky-rockets, &c.

LANTHORNS, Muſcovy, dark and common, uſed in the field when dark to light the gunners in the camp to prepare the ſtores.

LASHING RINGS, with hoops, fixed on the ſide-pieces of travelling-carriages to laſh the tarpauling that covers the gun, as alſo to tie the ſponge, rammer, and ladle.

LIMBER, a two-wheel carriage with ſhafts to faſten the trail of travelling carriages by means of the pintle or iron pin, when travelling, and taken off on the battery, or in the park of artillery, which is called unlimbering the guns.

LINCH-PIN, that which paſſes through the ends of the arms of an axle-tree, to keep the wheels or trucks from ſlipping out in travelling.

LINCH CLOUT, the flat iron under the ends of the arms of an axle-tree to ſtrengthen them, and to diminiſh the friction of the wheels.

LINE of DIRECTION, formerly marked upon guns, by a ſhort point upon the muzzle and cavity on the baſe ring, to direct the eye in pointing the gun, but are left off at preſent for no ſubſtantial reaſon.

LINT STOCK, the ſtick uſed by the gunners to faſten the match whereby they fire the guns or mortars.

LOCKS, of various ſorts, common for lockers in travelling carriages, or for boxes containing ſhot and powder or cartridges.

LOCKER HINGES, ſerve to faſten the cover of the lockers in travelling carriages.

LOCKING-PLATES, thin flat pieces of iron nailed on the ſides of a field carriage where the wheels touch it in turning, to prevent the wearing the wood in thoſe places.

LOOP, in a ſhip carriage, made of iron, faſtened one on the front of the fore axle-tree, and two on each ſide, through which the ropes or tackle paſs, whereby the guns are moved backwards and forwards on board of ſhips.

MARLIN, tarred, ſkains, white; are long wreaths or lines of untwiſted hemp, dipped in pitch or tar, with which cables and other ropes at ſea are wrapped round to prevent their fretting or rubbing in the blocks or pullies through which they paſs.

They ſerve the ſame purpoſe in artillery upon ropes uſed for draught for rigging gins, uſually put up in ſmall parcels called ſkains.

MATCH,

MATCH, uſed in artillery for firing guns and mortars; it ſerves likewiſe to trace batteries inſtead of ropes.

MATROSS, properly an apprentice to a gunner, and has the leaſt pay of any ſoldier in the artillery.

MEASURES, POWDER, made of copper, holding from an ounce to eight or twelve pounds, and are very convenient in a ſiege when guns or mortars are loaded with looſe powder, eſpecially in ricochet firing.

MEDICINE CHEST, ſerves the ſurgeons to carry their medicines and inſtruments when in the field.

METAL, GUN, a compoſition of tin and copper; the moſt common proportion is, to a hundred pounds of copper twelve pounds of tin: the founders will ſometimes vary from this proportion according to the hardneſs of the copper.

MORTARS, made of braſs or iron, uſed in the land ſervice and at ſea, for throwing ſhells or carcaſſes; the land are ſhorter and lighter, and the chambers hold leſs powder.

They are diſtinguiſhed by the diameter of their bores; thus, a thirteen, ten, or eight inch mortar are thoſe whoſe diameter of their bores are thirteen, ten, or eight inches, the royal and coehorn excepted; the royal carries a ſhell whoſe diameter is 5.5 inches, and that of the coehorn whoſe diameter is 4.6 inches.

MORTAR, SEA, are longer to prevent the ſetting the rigings of the ſhips on fire, and heavier to leſſen the ſhock upon the ſhip when the mortar is fired: none but thirteen and ten inch mortars are uſed at ſea; and as theſe mortars are ſometimes fired at a great diſtance, the chamber of the firſt holds 32 pounds of powder, and that of the ſecond 12.5 pounds.

MOULDS, for caſting ſhot for guns, piſtols, muſkets, and carabines; the firſt are of iron uſed by the founders, and the others by the artillery in garriſon and the field.

MOULDS, of wood or braſs, uſed in laboratory works for filling and driving all ſorts of rockets, and cartridges; of many different ſizes.

NAILS of various ſorts are uſed in artillery.

1. GARNISH, in travelling carriages, have pointed heads finiſhed like diamonds, with a ſmall narrow neck; they ſerve to faſten the plates with roſes, to cover the ſide-pieces from the ends of the trunnion-plates to five or ſix inches beyond the center of the carriage.
2. DIAMOND-HEADED, ſmall nails whoſe heads are made like a flat diamond, and ſerve to fix the plates upon travelling carriages, beſides the garniſh nails, both upon the trail, and thoſe at the head are near the trunnion-plates.
3. ROSE-BUDS are ſmall round-headed nails drove in the center of the roſes of the plates.
4. COUNTER-SUNK, thoſe that have flat round heads ſunk into the iron plates ſo as to be even with the outſide of it.
5. STREAK-NAILS, are thoſe which faſten the ſtreaks to the fellows of the wheels.
6. BOX-PINS, ſmall nails without heads to pin the nave-boxes to the naves.
7. STUBS are driven on the outſide of the nave hoops to keep them in their places.
8. FLAT-HEADED NAILS to faſten the locker or any hinges.
9. DOG NAILS have flat round heads, and one part of the ſhank next to the head is alſo round.

NAVE, that part of a wheel in which the arms of the axle-tree moves, and in which the ſpokes are drove and ſupported.

NAVE HOOPS are flat iron rings to bind the nave; there are generally three on each nave.

NAVE BOXES, there are two, one at each end, to diminiſh the friction of the axle-tree againſt the nave: they were formerly made of braſs, but experience has ſhewn that the caſt iron cauſe leſs friction and are much cheaper.

OGEE,

OGEE, an ornamental moulding in the ſhape of an S, taken from architecture and uſed in guns, mortars, and howitzes.

ORDNANCE, a name given to all that concerns artillery; thus, the commander in chief is called Maſter General of the Ordnance inſtead of artillery; the Lieutenant General of the Ordnance.

ORDNANCE, Board of, conſiſts of four officers, the ſurveyor general, clerk of the ordnance, ſtore-keeper, and the clerk of deliveries, over which preſides the maſter, or, in his abſence, the lieutenant-general: this board deliberates, regulates, and orders every thing relating to the artillery.

Guns, mortars, and howitzes are often called ordnance.

PAILS, made of wood bound with iron hoops and handles, hold generally four gallons, and ſerve in the field to fetch water for the uſe of artillery works.

PANNELS are the carriages which carry mortars and their beds upon a march.

PETARD, an engine to burſt open the gates of ſmall fortreſſes, is made of gun metal, fixed upon a board two inches thick and about two feet ſquare, to which it is ſcrewed, and holds from nine to twenty pounds of powder, with a hole at the end oppoſite to the plank to fill it, into which the vent is ſcrewed; the petard thus prepared is hung againſt the gate by means of a hook, or ſupported by three ſtaves faſtened to the plank, when fired burſts open the gate.

PICKETS, for ſeveral uſes; thoſe for pinning the faſcines of a battery, are from three to five feet long, the heads two or three inches diameter.

PICKETS, uſed in the park of artillery, are about ſix or eight inches long, ſhod with iron, to pin the park line.

PICKET, an out-guard poſted before an army, to give notice if an enemy approaches.

PINTLE, a long iron bolt fixed upon the middle of the limber bolſter, to go through the hole made into the

trail tranſom of a field carriage, when it is to be tranſported from one place to another.

PINTLE-PLATE, is a flat iron through which the pintle paſſes, and nailed to both ſides of the bolſter, nailed with eight diamond-headed nails.

PINTLE-WASHER, an iron ring through which the pintle paſſes, placed cloſe to the bolſter for the trail to move upon.

PINTLE-HOLE, is of an oval figure made in the trail tranſom of a field carriage, wider above than below, to leave room for the pintle to play in an uneven road.

PINTLE-PLATES, one above and one below the pintle hole, with a ring at the end, nailed to the trail tranſom with diamond-headed nails.

PIN, an iron nail or bolt, with a round head, and generally has a hole at the end to receive a key; there are many ſorts, as axle-tree pins or bolts, bolſter pins, pole pins, ſwing-tree pins, &c.

PLATFORMS, of a battery, is the bed of wood upon which the guns ſtand; each conſiſting of eighteen planks of oak or elm, a foot broad, two inches and a half thick, and from eight to fifteen feet long, nailed or pinn'd on five beams of about four inches ſquare, called ſleepers.

The platform muſt be higher behind than before by ſix or nine inches, to prevent too great a recoil, and to advance the gun eaſily when loaded.

POLE, in a four wheel carriage, is faſtened to the middle of the hind axle-tree, and paſſes between the fore-axle-tree and its bolſter, faſtened with the pole pin ſo as to move about it, and faſtens the fore and hind carriages together.

POINT-BLANK, is ſaid of a gun when laid level; and point-blank range the diſtance the ſhot goes upon a level plain.

PONTOON, a flat-bottomed boat, whoſe carcaſs of wood is lined within and without with tin; they ſerve to lay bridges over rivers for the artillery and army.

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The French pontoons are made of copper, and only on the outſide; though they coſt more at firſt, yet they laſt much longer than thoſe of tin, and when wore out the copper ſells nearly for as much as it coſt at firſt.

PONTOON CARRIAGE, made of two wheels and two long ſide-pieces, whoſe fore ends are ſupported by a limber, and ſerves to carry the pontoon, boards, and the pieces of timber laid acroſs for making the bridge.

PORT FIRE, a compoſition put in a paper caſe to fire guns and mortars upon particular occaſions, or to light recreative fire-works.

POUNDER, ſaid of guns to ſpecify a certain caliber; thus, a twenty-four, twelve, or a ſix-pounder are thoſe pieces whoſe balls weigh twenty-four, twelve, or ſix pounds.

POWDER, GUN, a compoſition of ſaltpetre, ſulphur, and charcoal; the uſual proportion is, to ſix of ſaltpetre, one of ſulphur, and one of charcoal, well mixed and grained, and is put into barrels which contain 100 pounds each, for the uſe of artillery.

POWDER-CART, a two-wheel carriage, covered with an angular roof of boards, and to prevent the powder from dampneſs a tarred canvaſs is put over the roof; on each ſide are lockers to hold ſhot in proportion to the quantity of powder, which is generally four barrels.

The French uſe powder-waggons, which carry 12 barrels of 200 pounds each, but then the ſhots are carried by themſelves on other carriages.

POWDER MAGAZINE, a building to hold the powder in fortified places, containing ſeveral rows of barrels laid one over another, and arched over ſo as to be bomb-proof.

POWDER MILL, where the materials are beat, mixed together, and grained; they are placed near rivers, and as far from any houſe as can be, for fear of accidents, which often happen.

PROOF

PROOF of FIRE ARMS, the general rule for proving guns and mortars is to fire them three times with double the quantity of powder as they are loaded with in common ſervice; but the rule of the ordnance is, that all guns under 24 pounders are loaded with powder as much as their ſhot weighs; a braſs 24 pounder with 21lb. a braſs 32 pounder with 26lb. 12 ounces, and a 42 pounder with 31 lb. 8 ounces; the iron 24 pounder with 18 lb. the 32 with 21 lb. 8 ounces, and the 42 with 25 lb.

The braſs light field-pieces are proved with powder that weighs half that of their ſhot, except in the 24 pounder, which is loaded with 10 lb. only.

The government allows 11 bullets of lead in the pound for the proof of muſkets, and 14.5, or 29 in two pounds, for ſervice; 17 in the pound for the proof of carabines, and 20 for ſervice; 28 in the pound for the proof of piſtols, and 34 for ſervice.

When guns of a new metal or of lighter conſtruction are proved, then, beſides the common proof, they are fired two or three hundred times as quick as they can be, loaded with the common charge given in ſervice. Our light ſix pounders were fired 300 times in three hours and twenty-ſeven minutes, loaded with 1 lb. and 4 ounces of powder.

QUADRANT, an inſtrument made of braſs or wood, divided into degrees, and each degree into ten parts, to lay guns or mortars to any angle of elevation.

The common ſort is a quadrant, one of the radii projects the quadrant of about a foot, and a plummet ſuſpended in its center, by means of a fine thread of ſilk, ſo that when the long end is introduced into the piece the plummet ſhows its elevation.

The beſt ſort has a ſpiral level fixed to a braſs radius, ſo when the long end is introduced into the piece, this radius is turned about its center till it is level, then its end ſhews the angle of elevation, or the inclination

clination from the horizon, whereas the firſt ſhews that angle from the vertical.

QUICK MATCH, made of three cotton ſtrands drawn into lengths, and put into a kettle juſt covered with white wine vinegar, and then a quantity of ſaltpetre and mealed powder is put in it, and boiled till well mixed. Others put only ſaltpetre into water, and after that take it out hot and lay it into a trough with ſome mealed powder, moiſtened with ſome ſpirits of wine, thoroughly wrought into the cotton by rolling it backwards and forwards with the hands; and when this is done, they are taken out ſeparately, drawn through mealed powder, and dried upon a line.

RANGE, the diſtance from the battery to the point where the ſhot or ſhell touches the ground.

RANGE, POINT BLANK, that when the piece lies in a horizontal direction and upon a level plane.

RANGE, RANDOM, when the piece is elevated at an angle of elevation of 45 degrees upon a level plane.

RAVES, are the upper wooden bars in a cart or waggon, ſupported by the round and flat ſtaves which enter into them.

REGIMENT OF ARTILLERY, is compoſed of three battalions, and (a company of cadets) each battalion of ten companies, and in time of war two companies of miners are added to it.

REINFORCE, the part of a gun next to the breech, which is made ſtronger to reſiſt the force of the powder: there are generally two, called the firſt and ſecond; the ſecond is ſomething ſmaller than the firſt, upon the ſuppoſition that when the powder is inflamed, and occupies a greater ſpace, its force is diminiſhed: we have ſhewn, in our conſtruction of guns, the abſurdity of this practice.

REINFORCE-RING; there are three in each gun, called the firſt, ſecond, and third; they are flat mouldings, like flat iron hoops, placed at the the breech end of

 the

the firſt and ſecond reinforce, projecting the reſt of the metal by about a quarter of an inch.

RELIEVER, an iron ring fixed to a handle by means of a ſocket, ſo as to be at right angles to it: it ſerves to diſengage the ſearcher of a gun, when one of its points are retained in a hole, and cannot be got out otherwiſe.

RIDER, a piece of wood ſomething higher than broad, the length equal to that of the body of the axletree upon which the ſide-pieces reſt in a four-wheel carriage, ſuch as the ammunition waggon, block carriage, and ſling waggons.

RINGS, of various uſes, ſuch as the laſhing rings in travelling carriages to laſh the ſponge, rammer, and ladle, as well as the tarpauling that covers the guns; the rings faſtened to the breeching bolts in ſhip-carriages; and the ſhaft-rings to faſten the harneſs of the ſhaft horſe by means of a wooden pin.

RIVETTING-PLATES, ſmall ſquare thin pieces of iron, through which the ends of bolts paſs, and rivetted upon them.

RODS, iron or wood, rammers, to drive the charges of muſkets, carabines, and piſtols home to the bottom of the pieces.

RODS, or STICKS, faſtened to ſky rockets, to make them riſe in a ſtrait line.

ROCKETS, made for recreative fire-works, both for the air and water; the ſky have ſticks fixed to them.

ROPES, of various lengths and thickneſs, according to the uſes they are made for; ſuch as drays, for the gin, for the ſling cart and waggon.

SALTPETRE, the principal ingredient for making gunpowder, is found in great plenty in ſome of the Eaſtern provinces and in Europe, and is extracted from the earth dug out of cellars and ſtables, or elſe near old walls where urine has been made.

SALTING BOXES, for holding mealed powder to throw upon the ſhells, when in the mortar, in order that

the fuzes may take fire from the blaſt of the powder; but it has been found that the fuze takes fire without mealed powder by which theſe boxes are rejected.

SAWS, for cutting wood, very uſeful in artillery, to cut planks of travelling carriages or platforms, and in general for all carpenters work, &c.

SCALES, of ſeveral ſorts, large and ſmall, to weigh from a grain to four or five tons.

SCALING LADDERS, uſed in ſcaling when a place is to be taken by ſurprize: they are made ſeveral ways; here we make them with flat ſtaves, ſo as they may move about their pins, and ſhut like a parallel ruler, for conveniently carrying them. The French make them of ſeveral pieces, ſo as to be joined together, and to be made of any neceſſary length; ſometimes they are made of ſingle ropes knotted at proper diſtances, with iron hooks at each end, one to faſten them upon the wall above, and the other in the ground; and ſometimes they are made with two ropes, and ſtaves between them to keep the ropes at a proper diſtance, and to tread upon.

SCREWS, faſtened to the caſcable of a light gun or howitz, by means of an iron bolt, and goes through a ſocket fixed upon the center tranſom, to lay the piece with inſtead of wedges, as is done in heavy pieces.

SEA MORTARS, are thoſe put on board of bomb ketches, to be uſed in the bombarding of ſea-ports; each ketch carries two, one of thirteen and one of ten, no leſs are uſed.

SETTERS, a round ſtick to drive fuzes, or any other compoſitions, into caſes made of paper.

SHAFT-RINGS, ſee rings.

SHAFTS of a carriage, are two poles joined together with croſs bars, whereby the hind horſe guides the carriage, and ſupports the forepart of the ſhafts, the hind turning round an iron bolt.

SHAFT BARS, are two pieces of wood to faſten the hind ends of the ſhafts together, into which they are pinned with nails of wood.

SHELLS, a hollow iron ball to throw out of mortars or howitzes, with a hole of about an inch diameter, to load them with powder and to receive the fuze; the bottom or part oppoſite the fuze is made heavier than the reſt, that the fuze may fall uppermoſt; but in ſmall elevations is not always the caſe, nor is it neceſſary; for when it falls firſt, it ſets fire to the powder in the ſhell. But whether it breaks or not, it would be better to make the ſhell every where of the ſame thickneſs, becauſe it would then burſt into a greater number of pieces than it does now.

SHEEP-SKINS, ſerve to cover the mortars or howitzes between firing, to prevent any wet or dampneſs getting in them.

SHOT-IRON, balls to load guns with; the ſhot of lead to load muſkets, carabines, or piſtols, which are called bullets.

SHOT-GRAPE, a certain number of ſmall ſhots of iron or lead, quilted together with canvas and ropes about a pin of iron or wood, fixed upon a bottom of the ſame matter, ſo as the whole together weigh nearly as much as the ſhot of that caliber.

SHOVEL, a miner's or pioneer's tool to dig and throw up earth.

SIDE-PIECES, of gun carriages, ſee carriages.

SIDE-STRAPS, in a field carriage, are flat iron bands which go round the ſide-pieces, in thoſe places where the wood is cut croſs the grain, to ſtrengthen them near the center and the trail.

SLEETS, are the parts of a mortar going from the chamber to the trunnions, to ſtrengthen that part, ſee plate II. fig. 5. by the letter n.

SLING-CART, or waggon, ſerves to carry guns or mortars a ſmall diſtance, by means of a rope faſtened with one end round the middle of the pieces, and wound upon a roller, by means of handſpikes, or a rackwork.

SOMMERS,

SOMMERS, in an ammunition waggon are the upper ſides, ſupported by the ſtaves entered into them with one of their ends and the other into the ſide-pieces.

SPIKES-HAND, is a leaver of four or five feet long, and about two and a half inches diameter, tapered on one end, and flat on the other, to be put under a gun or mortar to raiſe it from the ground, ſo as to put a ſkid or piece of wood under it.

SPONGE, of a gun or mortar, is a cylinder of wood fixed to a handle, and whoſe diameter and length are equal to the diameter of the ſhot in guns, covered with a lambſkin to clean the piece with after it has been fired.

SPOKES of a wheel, are the roundiſh parts between the nave and fellows, or circumference of the wheel.

STAPLES, are drove in the ſide-pieces of gun and mortar carriages, to faſten the keys of eye-bolts by means of chains; there are alſo ſtaples to the locker of gun carriages, and for ſeveral other uſes.

STAVES, round and flat, uſed in ammunition and other waggons or carts, the round and flat ſticks between the ſommers and ſide-pieces, or elſe in common or ſcaling ladders to tread upon.

STAYS, axle-tree, in truck carriages, are the irons which are fixed one end under the fore axle-tree, and the other to the ſide-pieces, in the form of an S.

STEEL-YARD, a ſort of ſcale whereby many things are weighed, by means of two weights only, one fixed near the end, and the other ſliding along the beam upon which the weights are marked, according to the diſtances from the point of ſuſpenſion.

STONE-MORTAR, made lighter than the common, with a cylindric ſpace near the chamber to hold a baſket filled with ſtones, which are flung amongſt an enemy, inſtead of ſhells or grapes, when near.

STOOL-BED, in a truck carriage, ſerves as a ſtool for the guns to lay upon, and the wedges for raiſing them.

STREAKES, are the iron bands on the outſide of a wheel to bind ſtrongly the fellows together.

STREAKE-

STREAKE NAILS, are thoſe drove through the ſtreakes into the fellows.

SULPHUR, or brimſtone, a mineral, very uſeful in making of powder, or in artificial fireworks.

SWEEP-BAR, of a waggon, is that which is fixed on the hind part of the fore-guide, and paſſes under the hind pole, which ſlides upon it.

SWING-TREE, the bar in a carriage faſtened acroſs the fore-guide, and to which the traces of the horſes are faſtened.

TABLES, a kind of regiſter to ſet down the dimenſions of carriages for guns, mortars, &c.

TARPAULINGS, are made of ſtrong canvas thoroughly tarred, and cut into different ſizes according to their ſeveral uſes in the field; ſuch as to cover the powder waggons, and tumbrels carrying ammunition from rain; each field-piece has likewiſe one to ſecure their ammunition boxes.

TENTS, made of canvas, for officers and ſoldiers to lay under when in the field.

TIN-TUBES, are uſed in guns to ſet fire to the powder inſtead of priming. Their advantage over priming is that they preſerve the vent from wearing.

TOMPEONS, or tampion, a wooden cylinder to put into the mouth of guns in travelling, to prevent the duſt or wet getting in.

Sometimes uſed to put into the chambers of mortars over the powder when the chamber is not full.

TONGS, of a waggon, a piece of wood fixed between the middle of the hind ends of the ſhafts, morticed into the fore croſs-bar and let into the hind-croſs-bar.

TOOLS, of all ſorts, for artificers, miners, laboratory, &c.

TRAIL, is the end of the travelling carriage oppoſite to the wheels, and upon which the carriage ſlides, when unlimbered, or upon the battery.

TRANSOM,

TRANSOM, the pieces of wood which join the ſide-pieces of gun carriages: there is but one in a truck carriage placed under the trunnion holes, and four in the wheel carriages; the trail, the center, the bed, and the breaſt tranſoms.

TRANSOM-PLATES, with hooks; there is one on each ſide of the ſide-pieces againſt each end of the tranſom, the bed-tranſom excepted, faſtened by two tranſom-bolts.

TRANSOM-BOLT, with burrs; they ſerve to tie the ſide-pieces to the tranſoms.

TRAVELLING CARRIAGES, ſee gun carriage.

TRAVELLING FORGE, ſee forge.

TRAVERSING PLATES, two thin iron plates nailed on the hind part of a truck carriage of guns, where the handſpike is uſed to traverſe the gun.

TRUCKS of a ſhip carriage, are wheels made of one piece of wood, from twelve to nineteen inches diameter, and their thickneſs is always equal to the caliber of the gun.

The trucks of garriſon carriages are made of caſt iron.

TRUCK CARRIAGE, goes upon four trucks of twenty-five inches diameter, has two flat ſide-pieces of ten inches broad, and ſerves to carry guns, ammunition, boxes, or any other weights from the ſtore-houſes to the water ſide, or to any ſmall diſtance.

TRUNNIONS, two cylindric pieces of metal in a gun or mortar which project the piece, and by which they are ſupported upon their carriages.

TRUNNION PLATES, are two plates in travelling carriages which cover the upper parts of the ſide-pieces, and goes under the trunnions, its fore-part ends with a flat roſe.

TUGPINS, are the iron pins which paſs through the fore ends of the ſhafts, to faſten the draught-chains for the fore horſes.

VENT, of all kinds of fire-arms, is a ſmall hole pierced at the end or near it of the bore or chamber, to prime the pieces with powder, or to introduce a tube, in order when lighted to ſet fire to the charge.

VENT-FIELD, is the part of a gun or howitz between the breech moulding and the aſtragal.

VENT-ASTRAGAL, that which determines the vent-field.

WADD, made of hay or ſtraw, and ſometimes of tow rolled up tight in a ball, and ſerves to be put into a gun after the powder, and rammed home, to prevent the powder from being ſcattered, which would have no effect when unconfined.

WADD-MILL, a hollow form of wood to make the wadds in of a proper ſize.

WADDING, hay, ſtraw, or any other forage generally carried along with the guns to be made into wadds.

WADD-HOOK, a ſtrong iron ſcrew, like thoſe that ſerve for drawing corks, mounted upon a wooden handle, to draw out the wadd, or any parts of cartridges, which often remain in guns, and when accumulated ſtops up the vent.

WASHER, a flat circular ring put on the axle-tree, between the linch-pin and ſmall end of the nave, to prevent the nave rubbing againſt the linch-pin and wearing it, as likewiſe to diminiſh the friction of the nave.

WEIGHTS, are of two ſorts, troy weight is the leaſt, and ſerves for all light goods, ſilver, gold, and medicine.

Averdupois ſerves for all heavy courſe commodities, and every thing is weighed by it in artillery.

WHEEL of a carriage, well known by every body; in artillery their ſtrength is always, or ſhould be, proportional to the weight they carry; the diameter of the wheels of heavy gun carriages is fifty-eight inches, and thoſe for light field-pieces fifty-two only.

FINIS.